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Multidimensional Inverse Problems and Completeness of the Products of Solutions to PDE

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A method is given for proving uniqueness theorems for some inverse problems. The method is based on a result on completeness of the products of solutions to PDE. As an example, the following uniqueness theorems are proved: (1) the scattering amplitude $A(\theta', \theta, k)$ known for all θ', $\theta \in S^2$ and a fixed $k>0$ determines the compactly supported $q(x) \in L^2(D)$ uniquely; (2) the surface data $u(x, y, k)$ known for all x, $y \in P := \{x : x_3 = 0\}$ and a fixed $k>0$ determine the compactly supported $v(x) \in L^2(D)$, $D \subset R^3_- := \{x : x_3 < 0\}$ uniquely. Here $[\nabla^2 + k^2 + k^2 v(x)]\, u(x, y, k) = -\delta(x-y)$ in R^3; (3) the surface data $u(x, y, k)$ known for all x, $y \in P$ and all $0 < k < k_0$, $k_0 > 0$ is arbitrarily small; determine $a_j(x)$, $j = 1, 2$, uniquely. Here $\nabla^2 u + k^2 u + k^2 a_1(x) + \nabla \cdot (a_2(x) \nabla u) = -\delta(x-y)$ in R^3, $a_1 \in L^2(D)$, $a_2 \in H^2(D)$; the same conclusion holds if the surface data are known at two distinct frequencies. (4) The surface data $u(x, y, k)$ known for all x, $y \in P$ and all $k > 0$ determine $v(x)$ and $h(k)$ uniquely. Here $[\nabla^2 + k^2 + k^2 v(x)]\, u = -\delta(x-y)\, h(k)$, $v(x) \in L^2(D)$, $h(k)$ is Fourier transform of a wavelet of compact support; (5) the conductivity $\sigma(x) \in W^{2,2}(D)$, $\sigma(x) \geqslant c > 0$, is uniquely determined by the measurements of u and σu_N on ∂D. Here N is the outward normal to ∂D, $D \subset R^3$ is a bounded domain with a smooth boundary ∂D, $\nabla \cdot (\sigma(x) \nabla u) = 0$ in D. (6) Necessary and sufficient conditions are given for a function $A(\theta', \theta, k)$, θ', $\theta \in S^2$, $k > 0$ is fixed, to be the scattering amplitude corresponding to a local potential from a certain class.

I. INTRODUCTION

For a long time the following inverse problems were not solved. They are of interest in many applications.

PROBLEM 1. Given the scattering amplitude $A(\theta', \theta, k)$ for all θ', $\theta \in S^2$ and a fixed $k > 0$, find the potential $q(x)$.

Here

$$[\nabla^2 + k^2 - q(x)]\, \psi(\theta, k, x) = 0 \qquad \text{in } R^3, \quad k > 0 \tag{1}$$

$$\psi(\theta, k, x) = \exp(ik\theta \cdot x) + v, \tag{2}$$

$$v = A_q(\theta', \theta, k)\, g(r) + o(r^{-1}) \quad \text{as} \quad r = |x| \to \infty, \quad \theta' = r^{-1}x, \tag{3}$$

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where $A_q(\theta', \theta, k) := A(\theta', \theta, k)$ is called the scattering amplitude, and

$$g(r) := r^{-1} \exp(ikr). \tag{4}$$

Throughout this paper we assume that $q(x) \in L^\infty(D)$ (the case $q(x) \in L^2(D)$ is treated in [18]), $q(x) = 0$ outside D, Im $q = 0$. Im stands for imaginary part. By D we will always denote a bounded domain in R^3.

PROBLEM 2. Given the surface data $u(x, y, k)$ for all x, $y \in P := \{x : x_3 = 0\}$ and a fixed $k > 0$, find the inhomogeneity $v(x)$ in the refraction coefficient.

Here

$$[\nabla^2 + k^2 + k^2 v(x)]\, u = -\delta(x - y) \qquad \text{in } R^3 \tag{5}$$

$$v(x) \in L^\infty(D), \qquad D \in R^3_- := \{x : x_3 < 0\} \tag{6}$$

$v(x) = 0$ outside D. (The case $v \in L^2(D)$ is treated in [18].)

PROBLEM 3. Given the surface data $u(x, y, k)$ for all x, $y \in P$ and all k in the interval $(0, k_0)$, where $k_0 > 0$ is an arbitrary small number, or for $k = k_j$, $j = 1, 2$, $0 < k_1 < k_2$, find $a_j(x)$, $j = 1, 2$.

Here

$$\nabla^2 u + k^2 u + k^2 a_1(x)\, u + \nabla \cdot (a_2(x) \nabla u) = -\delta(x - y) \qquad \text{in } R^3 \tag{7}$$

$$a_1(x) \in L^\infty(D), \qquad a_2(x) \in C^2(\bar{D}), \qquad 1 + a_2(x) > 0 \tag{8}$$

$D \in R^3_-$ and $\bar{D}$ is the closure of D. (The case $a_1 \in L^2(D)$, $a_2 \in H^2(D) := W^{2,2}(D)$ is treated in [18]. See also [5f]).

PROBLEM 4. Given the surface data $u(x, y, k)$ for all x, $y \in P$ and all $k > 0$ find $v(x)$ and $h(k)$.

Here

$$[\nabla^2 + k^2 + k^2 v(x)]\, u = -\delta(x - y)\, h(k), \tag{9}$$

$v(x)$ as in (6), and

$$h(k) = \int_0^T \exp(ikt)\, a(t)\, dt, \qquad \int_0^T |a|^2\, dt < \infty, \tag{10}$$

where $T > 0$ is a fixed number, Im $a = 0$.

PROBLEM 5. Let

$$\nabla \cdot (\sigma(x) \nabla u) = 0 \qquad \text{in } D \tag{11}$$

$$u = f, \qquad u_N = h \qquad \text{on} \quad \Gamma = \partial D. \tag{12}$$

Assume that Γ is smooth,

$$\sigma(x) \in W^{2,\infty}(D), \tag{13}$$

where $W^{l,p}(D)$ is the Sobolev space, N is the outward normal to Γ, and

$$0 < c \leqslant \sigma(x). \tag{14}$$

Here and below the c denote various positive constants.

The problem is: Given the set of pairs $\{f, \sigma h\}$, *where* f *runs through* $C^1(\Gamma)$, *find* $\sigma(x)$.

Note that if σ and f are known then u is uniquely determined by Eq. (11). Therefore $h = \sigma u_N$ is uniquely determined. We are interested in the inverse problem of finding $\sigma(x)$ from the boundary data $\{f, \sigma h\}$. The case $\sigma \in H^2(D)$ is treated in [18].

PROBLEM 6. Let $Lu = L_0 u + Vu$:

$$L_0 u = \sum_{|\alpha| = l} a_\alpha \partial^\alpha u(x), \qquad a_\alpha = \text{const.}, \quad x \in R^n, \tag{15}$$

$$Vu = \sum_{|\alpha| \leqslant m} b_\alpha(x)\, \partial^\alpha u, \qquad m < l. \tag{16}$$

Here α is a multi-index, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Define

$$N_D(L) := \{u : Lu = 0 \text{ in } D\}, \tag{17}$$

$$N_D^\infty(L) := \{u : u \in N_D(L) \cap C^\infty(\bar{D})\}, \tag{17'}$$

where $Lu = 0$ in the distribution sense. Let $f \in L^p(D)$, $p \geqslant 1$, and

$$\int_D fuw\, dx = 0 \qquad \text{for all} \quad u, w \in N_D(L). \tag{18}$$

The problem is: Under what assumptions on L does (18) *imply that* $f = 0$? *If it does, we say that L has property* $\mathscr{C}$ ($\mathscr{C}$ *for completeness of the set of products of solutions to homogeneous* PDE).

PROBLEM 7. What are the necessary and sufficient conditions for a function $A(\theta', \theta)$ to be the scattering amplitude at a fixed $k > 0$ corresponding to a potential $q(x) \in Q$, where Q is a given class of potentials?

For example, $q \in Q_D$, where

$$Q = \{q : |q| + |\nabla q| \leqslant c(1 + |x|)^{-a}, a > 3, \operatorname{Im} q = 0\},$$

and

$$Q_D = \{q : q \in Q \text{ and } q = 0 \text{ outside } D\}. \tag{19}$$

Let us briefly describe the significance of these problems and what is known about them.

The 3D (three-dimensional) inverse scattering problem in quantum mechanics has been studied mostly in the case when $A(\theta', \theta, k)$ is given for all θ', $\theta \in S^2$ and all $k > 0$. Uniqueness of its solution for these data has been known for a long time. In [1] one can find references on the subject. Characterization of the scattering data, that is, necessary and sufficient conditions for a function $A(\theta', \theta, k)$ given on $S^2 \times S^2 \times R_+$, $R_+ = (0, \infty)$, to be the scattering amplitude corresponding to a potential $q \in Q$ are first found in [2]. A partial characterization of the scattering data has been given in [1] and papers cited in [1]. By a partial characterization we mean a set of necessary conditions for $A(\theta', \theta, k)$ to be the scattering amplitude.

The uniqueness theorem for the solution to Problem 1 at fixed energy was not known, see [3, 26]. It is first obtained in [5b] as an application of the general method developed in [5a, b], in order to prove uniqueness theorems in multidimensional inverse scattering problems. This method has been developed by the author in the process of studying inverse problems of geophysics [4]. Problems 2–4 are typical examples of these problems. In [11] it is proved that the surface data $u(x, y, k)$ given for all x, $y \in P$ and all k, $0 < k < k_0$, where $k_0 > 0$ is an arbitrary small number, determine $v(x)$ in Eqs. (5)–(6) uniquely, and an analytical recovery of $v(x)$ from these data is given (see [4, Chap. 6] for this theory, its generalizations, and references). This was the first and still the only one exact solution to the 3D inverse problem of finding $v(x)$ from the surface data. No results were known about uniqueness of the solution to Problems 2, 3 inspite of the efforts of the people interested in the field. Problem 3 has been solved in the Born approximation in [4]. This solution does not provide any information concerning Problem 3 in exact formulation. Problem 4 is important in applications since the wavelet shape in geophysics is often not known because of the difficulties in modelling the exploration wave at the time of explosion. In (5), (7), (9), the function u describes an acoustic field generated by a point source located at y. The function $v(x)$ is the inhomogeneity in the refraction coefficient of a homogeneous medium. The case when the background is variable is considered in [10]. The functions

$a_j(x)$ in (7) describe variation in the density and bulk modulus of the medium. The function $h(k)$ in (9) is the Fourier transform (10) of the wavelet shape $a(t)$.

In this paper we prove uniqueness theorems for Problems 2 and 3 and give an analytical solution for Problem 4. In the presentation the short communications [5, b–e] are used, where the problems were solved for the first time. See also [18–26].

Problem 5 has been studied recently in [6, and 9]. In [9] a uniqueness of its solution has been proved for $\sigma(x)$ piecewise real analytic, while in [6a] uniqueness is proved for $\sigma \in C^\infty(\bar{D})$. In [6b] the problem of finding σ and its derivatives on Γ from the knowledge of the Dirichlet to Neumann map $\Lambda_\sigma : f \to \sigma h$ is considered. It is proved that one can uniquely determine σ and its derivatives on Γ from the above map. Moreover, some constructive formulas for recovery of σ, σ_N, etc. on Γ are given in terms of the expansion of the symbol of the pseudodifferential operator Λ_σ. However, no constructive formulas for recovery of σ inside D were given in the literature. Here we reduce the smoothness requirements on σ from $C^\infty(\bar{D})$ to $W^{2,\infty}(D)$ (to $W^{2,2}(D)$ in [18]) and give a constructive method to recover the Fourier transform of $\sigma(x)$ from the data $\{f, \sigma h\}$ (and to recover σ and σ_N on Γ, see [20]). Our method is simple and is based on the ideas in [5a, b]. The method of [6] allows one to treat $\sigma \in W^{2,\infty}(D)$ but not $\sigma \in H^2(D) := W^{2,2}(D)$.

Problem 6 has been considered for the first time in [5a] in the case when $Lu = \sum_{|\alpha| \leqslant l} a_\alpha \partial^\alpha u(x)$, $x \in R^n$, $a_\alpha = \text{const}$. The uniqueness theorem in the problem of finding $v(x)$ from the data $u(x, y, k)$, x, $y \in P$, $0 < k < k_0$, has been reduced in [4] to the problem 6 with $Lu = \nabla^2 u$. In [4] and in the original work [11], the uniqueness theorem has been proved in the course of the analytical solution to the problem. In [5, a–d] a systematic method to prove uniqueness theorems in inverse scattering problems has been developed on the basis of the solution to Problem 6 for some particular operators L, namely $Lu = [\nabla^2 + k^2 + q(x)]\, u$ and $Lu = \nabla^2 u + k^2 a_1(x)\, u + k^2 u + \nabla \cdot (a_2(x) \nabla u)$. This method and the method developed in [2] allow us to solve Problem 7. This problem has been open for a long time. Some known uniqueness theorems for inverse problems are not discussed here (see, e.g., [12]).

The rest of the paper is organized as follows. In Section II the results are formulated, in Sections III and IV the proofs are given, in Section V some open problems are formulated, some technical details are given, and additional results are proved.

In each section formulas are numbered autonomously. References to formulas in other sections are given in the form (II.8), for instance; that is formula (8) in Section II. References to formulas in the same section are given without the number of the section.

II. Formulation of the Results

Throughout the paper the notations and the assumptions on the unknown functions $q(x)$, $v(x)$, $a_j(x)$, $h(k)$, and $\sigma(x)$ are the same as in Section I and will not be repeated. The reader can consult the formulation of Problems 1–7 for these notations and assumptions.

THEOREM 1. *The knowledge of $A(\theta', \theta, k)$ for all θ', $\theta \in S^2$ at a fixed $k > 0$ determines the (compactly supported) $q(x)$, $q = \bar{q}$, uniquely.*

Remark 1. The conclusion of Theorem 1 remains valid if $A(\theta', \theta, k)$ is given for all $\theta' \in S_1^2$ and all $\theta \in S_2^2$, where S_j^2, $j = 1, 2$, are arbitrary open sets in S^2. Indeed, if $q(x)$ is compactly supported then $A(\theta', \theta, k)$ admits analytic continuation in θ and θ' on the variety $\theta \cdot \theta = 1$ in $\mathbb{C}^3$, where $\theta \cdot \theta := \theta_1^2 + \theta_2^2 + \theta_3^2$, $\theta \in \mathbb{C}^3$ [4]. Therefore if $A(\theta', \theta, k)$ is known on $S_1^2 \times S_2^2$ it is uniquely defined on $S^2 \times S^2$.

THEOREM 2. *The data $u(x, y, k)$ known for all x, $y \in P$ and a fixed $k > 0$ determine $v(x)$ uniquely.*

Remark 2. It is not known if the data $u(x, y, k)$ known for all $x \in P$ and all $y \in l := \{x : x_3 = x_2 = 0\}$ at a fixed $k > 0$ determine $v(x)$ uniquely.

THEOREM 3. *The data $u(x, y, k)$ known for all x, $y \in P$ and all $k \in (0, k_0)$ or for $k = k_j$, $j = 1, 2$, $0 < k_1 < k_2$, determine $a_1(x)$ and $a_2(x)$ uniquely.*

THEOREM 4. *The data $u(x, y, k)$ known for all x, $y \in P$ and all $k > 0$ determine $v(x)$ and $h(k)$ uniquely.*

THEOREM 5. *The set $\{f, \sigma h\}$, where f runs through all of $C^1(\Gamma)$ determines $\sigma(x) \in W^{2,2}(D)$ uniquely.*

PROPOSITION 5. *The set $\{f, h\}$ $\forall f \in C^1(\Gamma)$ determines $\sigma(x)$ uniquely up to a constant factor. The data $\{f, h\}$ $\forall f \in C^1(\Gamma)$ and $\sigma(s_0)$, where $s_0 \in \Gamma$ is an arbitrary fixed point, determine $\sigma(x)$ uniquely, $\sigma \in H^2(D)$.*

Proposition 5 is proved in [20]. It is more difficult to formulate results related to Problems 6 and 7. One needs some preparation in order to formulate the results.

Let us start with the results related to Problem 6 and use the ideas in [5a, b]. Consider first the case of operators with constant coefficients. Let

$$Lu = \sum_{|\alpha| \leqslant l} a_\alpha \partial^\alpha u(x), \qquad x \in R^n, \quad a_\alpha = \bar{a}_\alpha = \text{const.} \tag{1}$$

Define an algebraic variety M in $\mathbb{C}^n$ by the formula

$$M = \left\{ z : z \in \mathbb{C}^n, \sum_{|\alpha| \leqslant l} a_\alpha z^\alpha = 0 \right\}. \tag{2}$$

Let us introduce.

Condition A. There exist at least two points m_1 and m_2 on M such that the tangent planes T_{m1} and T_{m2} to M at these points are not parallel.

Condition A holds iff the following condition A′ does not hold.

Condition A′. The polynomial $\sum_{|\alpha| \leqslant l} a_\alpha z^\alpha$ is of the form

$$\left(b_0 + \sum_{j=1}^{n} b_j z_j \right)^l, \qquad b_j = \text{const}, \quad 0 \leqslant j \leqslant n.$$

Remark 3. Condition A may hold for L but fail to hold for L_0, the principal part of L. Example: $L_0 = (\sum_{j=1}^n b_j \partial_j)^2$, $L = L_0 + \partial_1 - \partial_2$.

THEOREM 6. *Assume that L is given by* (1), *condition* A *holds*, $f \in L^p(D)$, $p \geqslant 1$, *and*

$$\int_D fuw \, dx = 0 \qquad \textit{for all} \quad u, w \in N_D^\infty(L) \tag{3}$$

Then $f = 0$. *Conversely, if* (3) *implies that* $f = 0$ *for any* $f \in L^p(D)$, $p \geqslant 1$, *and* $a_\alpha = \bar{a}_\alpha$, *then condition* A *holds.*

Let L and L_1 be differential expressions (operators) with constant coefficients. Assume that

$$\int_D fuw \, dx = 0 \qquad \text{for all} \quad u \in N_D^\infty(L) \quad \text{and} \quad w \in N_D^\infty(L_1) \tag{4}$$

We will assume in what follows, that $f \in L^p(D)$, $p \geqslant 1$, and will not repeat this assumption anymore. Define for L_1 the set M_1 by formula (2) with $a_\alpha^{(1)}$ in place of a_α, where $a_\alpha^{(1)}$ are the constant coefficients of the operator L_1. Let us introduce

Condition B. There exist at least two points $m \in M$ and $m_1 \in M_1$ such that rank $(T_m, T_{m_1}) = n$. Here (T_m, T_{m_1}) denotes the union of the basis vectors of the tangent planes T_m and T_{m_1}.

THEOREM 7. *If condition B holds then* (4) *implies that* $f=0$.

Remark 4. The operator L in Theorem 6 and the operators L and L_1 in Theorem 7 are not necessarily elliptic; they can be hyperbolic, parabolic, or general operators with constant coefficients.

Let us now consider the case when

$$L_q := L = L_0 + V, \qquad L_0 u = \nabla^2 u, \qquad Vu = k^2 u - q(x)\, u, \tag{5}$$

$$q(x) \in L^\infty(D), \qquad D \subset R^3,\ q=0 \text{ in } \Omega = R^3 \setminus D \tag{6}$$

In Theorems 8 and 9 the operators L and L_0 are defined by (5) and (6).

THEOREM 8. *If*

$$\int_D fuw\, dx = 0 \qquad \text{for all } u,\ w \in N_D(L) \tag{7}$$

then $f=0$.

Let $q_j(x) \in L^\infty(D)$, $q_j(x)=0$ outside $D, j=1, 2$.

THEOREM 9. *If*

$$\int_D fuw\, dx = 0 \qquad \text{for all} \quad u \in N_D(L) \quad \text{and} \quad w \in N_D(L_0) \tag{8}$$

then $f=0$. *The same conclusion holds if* (8) *holds* $\forall u \in N_D(L_{q_1})$ *and* $\forall w \in N_D(L_{q_2})$, *and we say that the pair* (L_{q_1}, L_{q_2}) *has property* $\mathscr{C}$.

COROLLARY. *The knowledge of the integrals* $\int_D fuw\, dx\ \forall u \in N_D(L_{q_1})$, $\forall w \in N_D(L_{q_2})$ *determines* f *uniquely.*

Remark 5. The conclusion of Theorem 8 remains valid if u and w run through linear subsets of $N_D(L)$ which is dense in $N_D(L)$ in any norm such that $uw \in L^{p'}(D)$, where $p' = p(p-1)^{-1}$ for $p>1$ and $p'=\infty$ for $p=1$. A similar remark is valid for Theorem 9, where w may run through $N_D^\infty(L_0)$ and u through a dense subset of $N_D(L)$ in $L^{p'}(D)$ norm.

In order to formulate the solution to Problem 7 let us recall some basic facts in scattering theory (see, e.g., [1 or 4]). The starting point is the well-known relation

$$\psi(\theta, k, x) = \int_{S^2} S(\theta', \theta, k)\, \psi(-\theta', -k, x)\, d\theta' \tag{9}$$

where ψ is defined in (I.1)–(I.4), and

$$S(\theta', \theta, k) = \delta(\theta' - \theta) + \frac{ik}{2\pi} A(\theta', \theta, k). \tag{10}$$

The kernel $S(\theta', \theta, k)$ is called the S-matrix, and $\delta(\theta' - \theta)$ is the delta function. Let us write (9) as

$$\overline{v(\theta, k, x)} = v(-\theta, -k, x) + \frac{ik}{2\pi} \int_{S^2} A(\theta', \theta, k)\, v(-\theta' - k, x)\, d\theta' + \frac{ik}{2\pi} \int_{S^2} A(\theta', \theta, k) \exp(ik\theta' \cdot x)\, d\theta', \qquad \forall x \in R^3. \tag{11}$$

Here v is defined by (I.2). If $A(\theta', \theta) := A(\theta', \theta, k)$ for a fixed $k > 0$ is the scattering amplitude corresponding to the potential $q(x) \in Q_D$ then the following Condition C holds.

Condition C. Equation (11) for any $x \in R^3$ has a solution v such that the function ψ defined by (I.2) solves Eq. (I.1), conditions (I.3) and (I.4) hold, and formula

$$q(x) := \psi^{-1}(\nabla^2 + k^2)\, \psi, \qquad q(x) \in Q_D \tag{12}$$

defines the potential $q(x) \in Q_D$.

Therefore, *Condition C is a necessary condition* for a given function $A(\theta', \theta)$ to be the scattering amplitude at a fixed $k > 0$ corresponding to a potential $q(x) \in Q_D$. Our basic result is that *Condition C is also a sufficient condition for $A \in \mathscr{A}$, that is, for $A(\theta', \theta)$ to be the scattering amplitude at a fixed $k > 0$ corresponding to a $q(x) \in Q_D$.*

THEOREM 10. *$A \in \mathscr{A}$ iff condition C holds. Moreover, if Condition C holds then the coefficient $A_q(\theta', \theta, k)$ in the asymptotics* (I.3) *is equal to the given function $A(\theta', \theta) = A(\theta', \theta, k)$ which is the kernel in Eq.* (11).

Note that the necessity part: $A \in \mathscr{A} \Rightarrow C$ has already been established. The sufficiency part: $C \Rightarrow A \in \mathscr{A}$ says that if Eq. (11) (in which $A(\theta', \theta, k) = A(\theta', \theta)$, where $A(\theta', \theta)$ is the given function) has a solution $v(\theta, k, x)$ such that the function ψ defined by formula (I.2) solves Eq. (I.1) and has asymptotics (I.3), and $q(x)$ defined by (12) belongs to Q_D, then:

(i) Equation (11) has at most one solution with the above properties,

(ii) the coefficient A_q in (I.3) is identical to the given function $A(\theta', \theta)$.

Thus, the function $q(x)$ given by formula (12) is the potential $q(x) \in Q_D$ which produces the scattering amplitude equal to the given function $A(\theta', \theta)$ at the given value of $k > 0$. Uniqueness of this potential follows from Theorem 1. The condition $q \in Q_D$ is needed only because in the class Q_D the uniqueness theorem is established (this theorem follows from Theorem 1). Our argument in the proof of Theorem 10 will show that as Q

one can take any class of potentials for which the uniqueness theorem holds, in other words, for which the knowledge of $A(\theta', \theta, k)$ for a fixed k and all $\theta', \theta \in S^2$ determines $q \in Q$ uniquely.

Let us formulate a result which will be used often in proofs. First a similar result for $k=0$ was given in [6a]. In [5b] a simple and short proof of Theorem 11 is given. The method of [6a] does not apply to $q \in L^2(D)$. This case is handled in [18].

THEOREM 11. *If $q \in L^\infty(D)$, $D \subset R^3$ is a bounded domain, $q(x)=0$ outside D, then, for any fixed $k \geqslant 0$, the equation*

$$[\nabla^2 + k^2 - q(x)]\, u = 0 \qquad \text{in } R^3 \tag{13}$$

has a solution of the form

$$u(x, z) = \exp(iz \cdot x)[1 + R(x, z)], \qquad z \in \mathbb{C}^3, \quad z \cdot z = k^2, \tag{14}$$

where

$$\| R(x, z)\|_{L^2(D_1)} \leqslant c(1 + |z|)^{-1/2} \qquad \text{as} \quad |z| \to \infty, \quad \operatorname{Im} z \neq 0,\ z \cdot z = k^2 \tag{15}$$

uniformly in D_1 running through any bounded domain in R^3. The constant $c > 0$ in (15) does not depend on z but depends on D_1 and on $\|q\|_{L^\infty(D)}$. If $q \in W^{m,\infty}(D)$ then

$$\| R(x, z)\|_{H^m(D_1)} \leqslant c(1 + |z|)^{-1/2} \qquad \text{as} \quad |z| \to \infty, \quad \operatorname{Im} z \neq 0. \tag{16}$$

Here $H^m(D) = W^{m,2}(D)$ is the Sobolev space. The solution (14) is unique in $L^2(R^3, (1+|x|)^{-a})$ if $a < 2$.

Finally let us formulate

THEOREM 12. *Let $q \in L^\infty(D)$, $D \subset R^3$ is a bounded domain with a smooth boundary Γ. Let*

$$[\nabla^2 + k^2 - q(x)]\, u = 0 \qquad \text{in } D, \quad u = f, \quad u_N = h \text{ on } \Gamma, \tag{17}$$

$k \geqslant 0$ is a number, and k^2 is not an eigenvalue of the Dirichlet operator $-\nabla^2 + q(x)$ in D, N is the outward normal to Γ. Then, the knowledge of the set $\{f, h\}$ with f running through $C^1(\Gamma)$ determines $q(x)$ uniquely.

III. PROOFS

1. We first prove Theorems 6–9 since they are used in other proofs. We need

LEMMA 1. *Let $M \subset \mathbb{C}^n$ be an algebraic variety of (complex) dimension*

$n-1$. *If Condition* A *holds then for any* $\varepsilon>0$, *the set* $\{x_1+x_2\}$ *contains a ball* $B_\delta(m_1+m_2)\subset\mathbb{C}^n$. *Here* $B_\delta(m):=\{z:z\in\mathbb{C}^n,\ |z-m|<\delta\}$, x_j *runs through* $M\cap B_\varepsilon(m_j)$, $j=1,2$, *and* $\delta=\delta(\varepsilon)>0$.

Proof. The mapping $f:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ given by the formula $f(x,y)=x+y$ is linear and therefore df, its differential, at any point acts like the mapping itself. The restriction of df on $M\times M$ is defined on $T_{m_1}\times T_{m_2}$. If T_{m_1} is not parallel to T_{m_2} then the set $\{x_1+x_2\}$ contains a ball $B_\delta(m_1+m_2)$ if x_j runs through $T_{m_j}\cap B_\varepsilon(m_j)$. If $\varepsilon>0$ us sufficiently small the elements of $T_{m_j}\cap B_\varepsilon(m_j)$ differ very little from the elements of $M\cap B_\varepsilon(m_j)$. Therefore the conclusion of Lemma 1 follows.

LEMMA 2. *If Condition* B *holds then the set* $\{x+x_1\}$ *contains a ball* $B_\delta(m+m_1)\subset\mathbb{C}^n$. *Here* x *and* x_1 *run through* $M\cap B_\varepsilon(m)$ *and* $M_1\cap B_\varepsilon(m_1)$, *respectively.*

The proof of Lemma 2 is similar to the proof of Lemma 1 and is therefore omitted.

Proof of Theorem 6. *Sufficiency.* Assume that Condition A holds. Note that if $u=\exp(z\cdot x)$, $z\in M$, then $u\in N_D^\infty(L)$ for any D. Let $z_j\in M, j=1,2$, and take in (3) $u=\exp(z_1\cdot x)$ and $w=\exp(z_2\cdot x)$ to get

$$\int_D f(x)\exp\{(z_1+z_2)\cdot x\}\,dx=0. \tag{1}$$

By Lemma 1 the set $\{z_1+z_2\}$ contains a ball $B_\delta=B_\delta(m_1+m_2)$. Therefore (1) shows that

$$\int_D f(x)\exp(z\cdot x)\,dx=0 \qquad \text{for} \quad z\in B\subset\mathbb{C}^n. \tag{2}$$

Since D is bounded, the left side of (2) is an entire function of z which vanishes in a ball $B\subset\mathbb{C}^n$. By analytic continuation this function vanishes identically. This implies that $f(x)=0$. Therefore Condition A and (II.3) imply that $f=0$.

Necessity. Suppose that (II.3) implies that $f=0$ for any $f\in L^p(D)$, $p\geqslant 1$, but Condition A does not hold. Then Condition A′ holds. We wish to show that this is impossible. Assume for simplicity that $b_0=0$. The argument is similar if $b_0\neq 0$. If A′ holds with $b_0=0$ then $M=\{z:(\sum_{j=1}^n b_jz_j)=0\}$, $b_j=\text{const}$. The function $\exp(z\cdot x)$ belongs to $N_D^\infty(L)$ iff $z\in M$, that is $z\cdot b=0$, where $b=(b_1,\dots,b_n)\in R^n$. Let us make a rotation of the coordinate system $z\to\zeta$ so that in the new coordinate system $(\zeta_1,\dots,\zeta_n)$ we have $\zeta_n=\sum_{j=1}^n b_jz_j$. This means that we choose the coordinate system in which vector b is directed along the ζ_n axis. Then $M=\{\zeta:\zeta_n^l=0\}=\{\zeta:\zeta_n=0\}$. The corresponding coordinate transformation $x\to\xi$ brings the operator L into the form $L=\partial^l_{\xi_n}$. Therefore $N^\infty_D(L)=\{P_{l-1}(\xi_n)\}$, where $\xi:=(\xi', \xi_n)$, $\xi'=(\xi_1, ..., \xi_{n-1})$, $\xi \in R^n$ if $b \in R^n$. The general solution to the equation $Lu=\partial^l_{\xi_n}u=0$ is $u=P_{l-1}(\xi_n)$, where $P_{l-1}(\xi_n)$ is a polynomial in one variable of degree $l-1$ with arbitrary coefficients $\varphi_j(\xi')$, $0 \leqslant j \leqslant l-1$. If D' is the image of D under the coordinate transformation $x \to \xi$, then choose $f=h(\xi_n)\, d(\xi')$, $f \not\equiv 0$, such that the support f in D' is a box $a_j \leqslant \xi_j \leqslant b_j$, $1 \leqslant j \leqslant n$, $\int_{a_n}^{b_n} h(\xi_n)\, \xi_n^j\, d\xi_n=0$ for $0 \leqslant j \leqslant 2l-2$, $h(\xi_n)$ is continuous, and $d(\xi')$ is a continuous function vanishing for ξ' outside the set $a_j \leqslant \xi_j \leqslant b_j$, $1 \leqslant j \leqslant n-1$. Then (II.3) holds and $f \not\equiv 0$. Theorem 6 is proved.

Proof of Theorem 7. Similar to the proof of Theorem 6. Lemma 2 is used in place of Lemma 1. Details are left to the reader.

2. We give the Proofs of Theorems 8 and 9 assuming that Theorem 11 is valid, and then we prove Theorem 11. Proof of Theorem 8 requires a lemma of algebraic nature.

LEMMA 3. *Let $k \geqslant 0$ be given. Then there exist z and λ, $z \in \mathbb{C}^3$, $\lambda \in \mathbb{C}^3$, such that*

$$z \cdot z = k^2, \qquad \lambda \cdot \lambda = k^2, \qquad z-\lambda=p, \tag{3}$$

where $p \in R^3$ is an arbitrary given vector, and λ and z can be chosen so that $|\lambda|$ and $|z|$ are arbitrarily large. In particular, conditions

$$|\lambda| \to \infty, \qquad |z| \to \infty \tag{4}$$

can be satisfied.

Proof. Put $z=a+ib$, $\lambda=\alpha+i\beta$, where a, b, α, $\beta \in R^3$. The first two equations (3) can be written as

$$a^2-b^2=k^2, \qquad a \cdot b=0, \qquad \alpha^2-\beta^2=k^2, \qquad \alpha \cdot \beta=0, \tag{5}$$

while the last equation (3) takes the form

$$a=\alpha+p, \qquad b=\beta. \tag{6}$$

Note that (5) and (6) are 10 equations for 12 variables, coordinates of the vectors a, b, α, β. Without loss of generality let us assume that $p=e_3$, where e_j, $j=1, 2, 3$, are the unit vectors of the orthonormal basis in R^3. This amounts to choosing the third coordinate axis along vector p and the scale in such a way that $|p|=1$. Eliminate a and b using Eqs. (6) in (5):

$$(\alpha+p)^2-\beta^2=k^2, \qquad (\alpha+p) \cdot \beta=0, \qquad \alpha^2-\beta^2=k^2, \qquad \alpha \cdot \beta=0. \tag{7}$$

This and the equation $p=e_3$ imply that

$$2\alpha_3+1=0, \qquad \beta_3=0. \tag{8}$$

Put

$$\beta = c_2 e_2, \qquad \alpha = c_1 e_1 - \tfrac{1}{2} e_3, \tag{9}$$

where c_1 and c_2 choose so that

$$c_1^2 - c_2^2 = k^2 - \tfrac{1}{4}. \tag{10}$$

Then Eqs. (7) are satisfied for any real c_1 and c_2 which satisfy (10). Indeed, with the choice (9) one has $\alpha \cdot \beta = 0$, $(\alpha + e_3)\,\beta = 0$, $\alpha^2 - \beta^2 = c_1^2 + \frac{1}{4} - c_2^2 = k^2$, $(\alpha + e_3)^2 - \beta^2 = k^2$. With α and β given by (9) define a and b by (6). Then Eqs. (5) and (6) are satisfied for any real c_1 and c_2 which satisfy Eq. (10). Clearly one can choose $|c_1|$ and $|c_2|$ arbitrarily large and still satisfy (10) with any fixed $k \geqslant 0$. Therefore conditions (4) also can be satisfied; note that $|\lambda| = (\alpha^2 + \beta^2)^{1/2} = (c_1^2 + \frac{1}{4} + c_2^2)^{1/2} = |z|$. Lemma 3 is proved.

Proof of Theorem 8. Take z and λ such that (3) holds. Using Theorem 11, take u in (II.7) of the form (II.14) and w in (II.7) in the same form with $-\lambda$ in place of z. Then the assumption (II.7) takes the form

$$\int_D f \exp(ip \cdot x)(1 + \varepsilon)\, dx = 0, \tag{11}$$

where $\varepsilon \to 0$ as $|z| \to \infty$ and $|\lambda| \to \infty$. Using Lemma 3 and passing to the limit $|z| \to \infty$ and $|\lambda| \to \infty$, one obtains from (11) that

$$\int_D f \exp(ip \cdot x)\, dx = 0 \qquad \text{for any} \quad p \in R^3. \tag{12}$$

This implies that $f = 0$. Theorem 8 is proved.

Proof of Theorem 9. Similar to the proof of Theorem 8 and therefore omitted.

Let us pass to the proof of Theorem 11. Note that if one substitutes (II.14) into (II.13) then the resulting equation for R is

$$\nabla^2 R + 2iz \cdot \nabla R - q(x)\, R = q(x). \tag{13}$$

Let us first study the equation

$$\nabla^2 u + 2iz \cdot \nabla u = f \qquad \text{in } R^3, \tag{14}$$

assuming that $f \in L^2(D)$, $f = 0$ outside D, $z \cdot z = k^2$, and $\operatorname{Im} z \neq 0$.

LEMMA 4. *For all sufficiently large* $|z|$, $\operatorname{Im} z \neq 0$, *Eq.* (14) *has a solution* $u \in L^2_{\text{loc}}(R^3)$, *such that*

$$\|u\|_{L^2(D_1)} \leqslant c(1 + |z|)^{-1/2} \|f\|_{L^2(D)}, \qquad |z| \gg 1, \quad \operatorname{Im} z \neq 0, z \cdot z = k^2. \tag{15}$$

Here D_1 is an arbitrary bounded domain and $c>0$ is a constant which does not depend on z but depends on D_1.

Proof. Let

$$\tilde{u}(\lambda) := (2\pi)^{-3} \int \exp(-i\lambda\cdot x)\, u(x)\, dx, \qquad \int = \int_{R^3}. \tag{16}$$

Take the Fourier transform of (14) to get $\tilde{u} = -(\lambda^2 + 2\lambda\cdot z)^{-1}\tilde{f}$. Thus

$$u(x) = -\int (\lambda^2 + 2\lambda\cdot z)^{-1}\tilde{f}(\lambda)\exp(i\lambda\cdot x)\, d\lambda. \tag{17}$$

We wish to prove that the function (17) satisfies inequality (15). Let $z = a + ib$, a and $b \in R^3$, $b \neq 0$. Then, since $z\cdot z = k^2$, one has

$$a^2 - b^2 = k^2, \qquad a\cdot b = 0. \tag{18}$$

Let $a = \tau e_2, b = te_1$, where $t := |b| > 0$, $\tau := |a| = (t^2 + k^2)^{1/2} > 0$, and e_j, $1 \leqslant j \leqslant 3$, are the unit vectors of an orthonormal basis in R^3. One has

$$\begin{aligned}\lambda^2 + 2\lambda\cdot z &= \lambda_1^2 + 2it\,\lambda_1 + \lambda_2^2 + \lambda_3^2 + 2\tau\lambda_2 \\ &= \lambda_1^2 + 2it\lambda_1 + \lambda_3^2 + (\lambda_2+\tau)^2 - \tau^2.\end{aligned} \tag{19}$$

Function (19) vanishes iff

$$\lambda_1 = 0 \qquad \text{and} \qquad (\lambda_2+\tau)^2 + \lambda_3^2 = \tau^2. \tag{20}$$

Equation (20) defines a circle C_τ on the plane (λ_2, λ_3) centered at $(0, -\tau, 0)$ with radius τ. Let T_δ be the torus which is obtained by moving a square with the side 2δ centered at the points of C_τ and perpendicular to the plane (λ_2, λ_3). One has

$$\begin{aligned}|u(x)| \leqslant{}& \left|\int_{T_\delta} \tilde{f}(\lambda)(\lambda^2 + 2\lambda\cdot z)^{-1}\exp(i\lambda\cdot x)\, d\lambda\right| \\ &+ \left|\int_{R^3\setminus T_\delta} \tilde{f}(\lambda)(\lambda^2 + 2\lambda\cdot z)^{-1}\exp(i\lambda\cdot x)\, d\lambda\right| := |u_1| + |u_2|.\end{aligned} \tag{21}$$

Note that

$$|\lambda^2 + 2\lambda\cdot z| \geqslant 2t\delta \qquad \text{if} \quad \lambda \in R^3\setminus T_\delta. \tag{22}$$

Therefore, by Parseval's equality,

$$\begin{aligned}\|u_2\|_{L^2(R^3)} &= \left(\int_{R^3\setminus T_\delta} |\tilde{f}|^2\, |\lambda^2 + 2\lambda\cdot z|^{-2}\, d\lambda\right)^{1/2} \\ &\leqslant (2\delta t)^{-1}\, \|f\|_{L^2(R^3)} = (2\delta t)^{-1}\, \|f\|_{L^2(D)}.\end{aligned} \tag{23}$$

Furthermore,

$$
\begin{aligned}
|u_1(x)| &\leqslant \| \tilde{f} \|_{L^\infty(R^3)} \int_{T_\delta} |\lambda^2 + 2\lambda \cdot z|^{-1} \, d\lambda \\
&\leqslant 2\pi \, \| f \|_{L^1(D)} \int_{-\delta}^{\delta} d\lambda_1 \int_{\tau-\delta}^{\tau+\delta} \frac{r \, dr}{[4t^2\lambda_1^2 + (\lambda_1^2 + r^2 - \tau^2)^2]^{1/2}} \\
&\leqslant c \, \| f \|_{L^2(D)} \int_0^{\delta} d\lambda_1 \int_{\lambda_1^2+\delta^2-2\delta\tau}^{\lambda_1^2+\delta^2+2\delta\tau} (4t^2\lambda_1^2 + \mu^2)^{-1/2} \, d\mu \\
&\leqslant c \, \| f \|_{L^2(D)} \int_0^{\delta} d\lambda_1 \int_{-2\delta\tau}^{3\delta\tau} (4t^2\lambda_1^2 + \mu^2)^{-1/2} \, d\mu \\
&\leqslant c \, \| f \|_{L^2(D)} \int_0^{\delta} d\lambda_1 \int_0^{3\delta\tau} (4t^2\lambda_1^2 + \mu^2)^{-1/2} \, d\mu.
\end{aligned} \tag{24}
$$

Here and below $c > 0$ denote various constants; we used polar coordinates, the substitution $\mu = \lambda_1^2 + r^2 - \tau^2$, and the inequalities $\lambda_1^2 \leqslant \delta^2$, $\lambda_1^2 + \delta^2 \leqslant \delta\tau$, which hold if $\tau \geqslant 2$. This last inequality is valid since $\tau \to \infty$. Let $2t\lambda_1 = \beta$ in (24). Then the integral in the right side of (24) is not greater than

$$
(2t)^{-1} \int_0^{2t\delta} d\beta \int_0^{3\delta\tau} (\beta^2 + \mu^2)^{-1/2} \, d\mu \leqslant ct^{-1}\tau\delta \leqslant c\delta. \tag{25}
$$

Here we used polar coordinates and the inequality $\tau \geqslant t$. Combining (24) and (25) one gets

$$
\| u_1(x) \|_{L^\infty(R^3)} \leqslant c\delta \, \| f \|_{L^2(D)}. \tag{26}
$$

Thus

$$
\| u_1(x) \|_{L^2(D_1)} \leqslant c\delta \, \| f \|_{L^2(D)}, \qquad c = c(D_1), \tag{27}
$$

where D_1 is an arbitrary bounded domain. From (23) and (27) it follows that

$$
\| u \|_{L^2(D_1)} \leqslant c[\delta + (\delta t)^{-1}] \, \| f \|_{L^2(D)}; \tag{28}
$$

note that

$$
\min_{\delta > 0} [\delta + (\delta t)^{-1}] = 2t^{-1/2}. \tag{29}
$$

From (28) and (29) it follows that

$$
\| u \|_{L^2(D_1)} \leqslant ct^{-1/2} \, \| f \|_{L^2(D)}, \qquad t \gg 1. \tag{30}
$$

This estimate is equivalent to (15) since $c_1 t \leqslant |z| \leqslant c_2 t$, $c_1 > 0$, for all sufficiently large $|z|$. Lemma 4 is proved.

Estimates of $\|u\|_{L^\infty(D_1)}$ are given in [18]: $\|u\|_{L^\infty(D_1)} \leqslant ct^{-1/2} \ln t \, \|f\|_{L^2(D)}$ as $t \to +\infty$.

Next we consider the equation

$$\nabla^2 u + 2iz \cdot \nabla u - q(x)\, u = f \qquad \text{in } R^3 \tag{31}$$

where $q(x) \in L^\infty(D)$, $q = 0$ outside D.

LEMMA 5. *Under the assumptions of Lemma* 4 *Eq.* (31) *has a solution* $u \in L^2_{\text{loc}}(R^3)$ *and the inequality* (15) *holds for this solution of Eq.* (31).

Proof. Let $Lu := (\nabla^2 + 2iz \cdot \nabla)\, u$. Write (31) as

$$Lu = q(x)\, u + f. \tag{32}$$

If $u \in L^2_{\text{loc}}(R^3)$ solves (32) then $qu \in L^2(D)$, $qu = 0$ outside D since $q = 0$ outside D. By Lemma 4, one has

$$\begin{aligned} \|u\|_{L^2(D)} &\leqslant c\,|z|^{-1/2} (\|f\|_{L^2(D)} + \|qu\|_{L^2(D)}) \\ &\leqslant c\,|z|^{-1/2} (\|f\|_{L(D)} + \|q\|_{L^\infty(D)} \|u\|_{L^2(D)}). \end{aligned} \tag{33}$$

If $|z|$ is sufficiently large, (33) implies that

$$\|u\|_{L^2(D)} \leqslant c\,|z|^{-1/2} \|f\|_{L^2(D)}, \qquad |z| \gg 1. \tag{34}$$

In order to prove that Eq. (32) has a solution $u \in L^2_{\text{loc}}(R^3)$ note that for any bounded domain $D_1 \supset D$, $|z| \gg 1$, and $u \in L^2(D_1)$ we have

$$\|L^{-1} qu\|_{L^2(D_1)} \leqslant c\,|z|^{-1/2} \|q\|_{L^\infty(R^3)} \|u\|_{L^2(D_1)} \tag{35}$$

according to Lemma 4; $L^{-1}f$ is defined by (17). Therefore (32) can be written as

$$u = L^{-1}qu + L^{-1}f, \tag{36}$$

where the operator $L^{-1}q$ is a linear contraction in $L^2(D_1)$ for all sufficiently large $|z|$. Therefore Eq. (36) is uniquely solvable in $L^2(D_1)$ for all sufficiently large $|z|$, and its solution satisfies inequality (15). Lemma 5 is proved.

Under certain assumptions about the rate of decay at infinity of the solution constructed in Lemma 5 one can prove uniqueness of this solution. For example, if $u \in L^2_{\text{loc}}(R^3) \cap \mathscr{S}'$, where $\mathscr{S}'$ is the Schwartz class of distribution (temperate distributions), then one can use the following result.

PROPOSITION 1 [14, p. 174]. *Let* $u(x) \in L^2_{\text{loc}} \cap \mathscr{S}'$, $x \in R^n$, *and assume* *that its Fourier transform $\tilde{u}(\lambda)$ is supported by a C^1 manifold M of codimension p. If*

$$\limsup_{R \to \infty} R^{-p} \int_{|x| \leqslant R} |u(x)|^2 \, dx < \infty, \tag{37}$$

then $\tilde{u}$ is an L^2 density $\tilde{u}_0 \, ds$ on M, and

$$\int_M |\tilde{u}_0|^2 \, ds \leqslant c \limsup_{R \to \infty} R^{-p} \int_{|x| \leqslant R} |u|^2 \, dx. \tag{38}$$

Here $c > 0$ does not depend on R and u, ds is the Euclidean surface area of M.

From Proposition 1 the following uniqueness result follows immediately. Note that $L^2(R^3, p(x))$ is the weighted space with the norm $\|u\| := (\int |u|^2 p(x) \, dx)^{1/2}$, $p(x) > 0$.

LEMMA 6. *Equation (14) has at most one solution in the space $L_a^2 = L^2(R^3, (1 + |x|)^{-a})$ if $a < 2$.*

Proof. Since Eq. (14) is linear it suffices to prove that the homogeneous equaton (14) has only the trivial solution in L_a^2 if $a < 2$. If $u \in L_a^2$ then $u \in L_{\text{loc}}^2 \cap \mathscr{S}'$. Taking the Fourier transform of the homogeneous equation (14) yields

$$(\lambda^2 + 2\lambda \cdot z) \, \tilde{u} = 0. \tag{39}$$

Therefore $\tilde{u}$ is supported on the manifold C_τ given by Eq. (20). This is a smooth manifold of codimension $p = 2$ in R^3. If $u \in L_a^2$ then

$$\infty > \int_{R^3} (1 + |x|)^{-a} |u|^2 \, dx \geqslant (1 + R)^{-a} \int_{|x| \leqslant R} |u|^2 \, dx. \tag{40}$$

Therefore, if $a < 2$,

$$R^{-2} \int_{|x| \leqslant R} |u|^2 \, dx \leqslant c R^{a-2} \to 0 \qquad \text{as} \quad R \to \infty. \tag{41}$$

From (41) and (38) one concludes that $\tilde{u}_0 = 0$ and $u = 0$. Lemma 6 is proved.

The proof of Lemma 6 is essentially the same as the proof of Corollary 3.4 in [6a, p. 166]. Another argument is in Section V.6.

It is now easy to prove theorem 11.

Proof of Theorem 11. We have to prove that Eq. (13) has a solution which satisfies inequality (II.15). This follows immediately from Lemma 5. If $q \in W^{m,\infty}(D)$ then one can differentiate m times Eq. (13) and apply

Lemma 5 to the resulting equation. The last statement in Theorem 11 follows from Lemma 6. Theorem 11 is proved.

3. We now prove Theorem 1.

Proof of Theorem 1. The starting point is the well-known formula (see, e.g., [1 or 4]):

$$-4\pi A(\theta', \theta, k) = \int \exp(-ik\theta' \cdot x) \cdot q(x)\, \psi(\theta, k, x)\, dx. \tag{42}$$

Apply Theorem 11: take in (42) in place of ψ the special solution (II.14) $u(\theta, k, x) = \exp(ik\theta \cdot x)[1 + R(x, \theta)]$, where $\| R(x, \theta)\|_{L^2(D)} \to 0$ as $|\theta| \to \infty$, $\theta \in \mathbb{C}^3$, $\theta \cdot \theta = 1$ (see Section V.6 below). Take $\theta' \in \mathbb{C}^3$, $\theta' \cdot \theta' = 1$. The function $A(\theta', \theta, k)$ can be continued analytically on the subset of $\mathbb{C}^3 \times \mathbb{C}^3$ given by the equations

$$\theta \cdot \theta = 1, \qquad \theta' \cdot \theta' = 1. \tag{43}$$

The values of $A(\theta', \theta, k)$ for real θ and θ', that is, on $S^2 \times S^2$, determine uniquely $A(\theta', \theta, k)$ on the set (43) in $\mathbb{C}^3 \times \mathbb{C}^3$. For example, if one takes $\theta \in S^2$, $\theta = (\theta_1, \theta_2, (1 - \theta_1^2 - \theta_2^2)^{1/2})$, then A is analytic in θ_1, θ_2 in a neighbourhood of the origin in $\mathbb{C}^2$ and is known in a neighbourhood of the origin on R^2. By analytic continuation it is uniquely determined in a neighbourhood of the origin in $\mathbb{C}^2$. A similar argument applies to A as a function of θ'. Therefore the knowledge of $A(\theta', \theta, k)$ on $S^2 \times S^2$ determines $A(\theta', \theta, k)$ uniquely on the set (43) in $\mathbb{C}^3 \times \mathbb{C}^3$ ([4], p. 62).

Choose an arbitrary $p \in R^3$ and pick θ and θ' so that

$$k(\theta - \theta') = p, \qquad |\theta| \to \infty, \qquad |\theta'| \to \infty, \tag{44}$$

and conditions (43) hold. This is possible by Lemma 3. Then pass to the limit $|\theta| \to \infty$, $|\theta'| \to \infty$ in formula (42) keeping conditions (43) and (44) satisfied. The left side in (42) is a known quantity, so that in the limit one obtains the Fourier transform

$$\int q(x) \exp(ip \cdot x)\, dx := \tilde{q}(p) \tag{45}$$

of the potential. Therefore $q(x)$ is uniquely determined by taking the inverse Fourier transform of the function (45). Theorem 1 is proved.

Remark. Our basic idea is: if $A(\theta', \theta, k)$ is known for all $\theta', \theta \in S^2$ and a fixed $k > 0$, then, by (42), the set of integrals of $q(x)$ times functions from a complete in $L^2(D)$ set is known. This knowledge determines $q(x)$ uniquely. The set $\{\exp(-ik\theta' \cdot x)\, \psi(x, \theta, k)\}\ \forall \theta, \theta' \in S^2$, where ψ is defined in (I.1)–(II.3), is complete in $L^2(D)$, see Section V.6.

4. *Proof of Theorem* 2. We start with the equation

$$k^{-2}[u(x, y, k) - g(x, y, k)]$$
$$= \int g(x, \xi, k)\, v(\xi)\, u(\xi, y, k)\, d\xi, \qquad \forall x, y \in P, \tag{46}$$

where

$$g(x, y, k) = \frac{\exp(ik\,|x-y|)}{4\pi\,|x-y|}. \tag{47}$$

The left side in (46) is known. Let us denote it by $f(x, y)$. We do not show dependence on k since $k > 0$ is fixed. The integral in (46) is taken over the support of $v(x)$. Let us show that the data $f(x, y)$ uniquely determine the set of the integrals

$$\int v(\xi)\, u(\xi)\, w(\xi)\, d\xi \qquad \text{for all} \quad u \in N_D(\nabla^2 + k^2),$$
$$w \in N_D(\nabla^2 + k^2 + k^2 v(x)). \tag{48}$$

Indeed, multiply (46) by $\varphi(x) \in C_0^\infty(P)$ and integrate over P. Denote

$$u(\xi) := \int_P g(x, \xi, k)\, \varphi(x)\, dx; \tag{49}$$

note that $u \in N_D(\nabla^2 + k^2)$ since P does not intersect D, the support of v. Let us prove that the set $\{u\}$ given by (49) is dense in $N_D(\nabla^2 + k^2)$ (in $L^2(D)$) provided that $\varphi(x)$ in (49) runs through all of $C_0^\infty(P)$. It suffices to show that the set $\{u(s)\}$, $s \in \Gamma = \partial D$, is dense in $L^2(\Gamma)$. Suppose it is not. Then there is a function $h \in L^2(\Gamma)$ such that

$$0 = \int_\Gamma h(s) \int_P g(x, s, k)\, \varphi(x)\, dx\, ds \qquad \text{for all} \quad \varphi \in C_0^\infty(P). \tag{50}$$

This implies that the function

$$p(x) := \int_\Gamma g(x, s, k)\, h(s)\, ds \tag{51}$$

vanishes on P:

$$p(x) = 0 \qquad \text{for} \quad x \in P. \tag{52}$$

Since $p(x)$ satisfies in $R^3_+ = \{x : x_3 > 0\}$ the radiation condition, the equation

$$(\nabla^2 + k^2)\, p = 0 \qquad \text{in} \quad R^3_+, \tag{53}$$

and the boundary condition (52), one concludes that $p(x)=0$ in R^3_+ (see, e.g., [4]). By the unique continuation property for the solutions to (53) one concludes that $p(x)=0$ outside D and

$$p(x)=0 \qquad \text{on} \quad \Gamma. \tag{54}$$

Since

$$(\nabla^2+k^2)\,p=0 \qquad \text{in} \quad D, \tag{55}$$

one can conclude that $p=0$ in D provided that k^2 is not an eigenvalue of the Dirichlet Laplacian in D. Assuming this for a moment, we then derive that $h(s)=0$ by the jump relation for the normal derivative of the potential of the simple layer (51). If k^2 is an eigenvalue of the Dirichlet Laplacian in D then we take $D_1 \supset D$, $D_1 \subset R^3_-$, such that k^2 is not the eigenvalue of the Dirichlet Laplacian in D_1 and repeat our argument. The argument then proves the density of the values $\{u(s)\}$, $s \in \Gamma_1$, in $L^2(\Gamma_1)$, which is sufficient for our purposes. If the boundary values on Γ (or on Γ_1) of the set $\{u\}$ are dense in $L^2(\Gamma)$ then an arbitrary element $u \in N_D(\nabla^2+k^2)$ can be approximated in $H^{1/2}(D)$ by the functions from the set (49). Actually, the values of $u(\xi)$ on Γ are smooth functions and one can approximate smooth elements of $N_D(\nabla^2+k^2)$ smoothly (see Sections V.6, V.12).

A similar argument applies to the variable y in (46) and as a result we conclude that the set of integrals (48) is known. Theorem 9 is not applicable directly to the set (48) since w depend on v. Therefore we use Theorem 11 and take as w, solutions of the form (II.14), and take $u=\exp(-i\lambda\cdot x)$, $\lambda\cdot\lambda=k^2$, $\lambda\in\mathbb{C}^3$. We then use Lemma 3 and choose λ and z so that conditions (3) and (4) hold. Passing to the limit $|\lambda|\to\infty$, $|z|\to\infty$, in (48) gives the Fourier transform of v: $\int v(\xi)\exp(ip\cdot\xi)\,d\xi$. Therefore $v(x)$ is uniquely determined. Theorem 2 is proved.

5. *Proof of Theorem* 3. We give a proof for the data given for $0<k<k_0$. A proof for the data given at two distinct frequencies is in [5f]. See also [5c, d]. It has been proved in [4, 13] that the solution to Eq. (I.7) has a limit as $k\to 0$ provided that the limiting problem

$$\nabla^2 u+\nabla\cdot(a_2(x)\,\nabla u)=-\delta(x-y) \qquad \text{in} \quad R^3,\ u(\infty)=0 \tag{56}$$

has at most one solution. If $1+a_2(x)>0$ (see (I.8)) then (56) has at most one solution. Indeed, multiply the homogeneous equation (56) by $u(x)$ and integrate by parts. Note that since a_2 is real-valued one can assume that $u(x)$ is real-valued. The resulting identity shows that $u=0$. Therefore, according to [4], there exists the limit (in H^2_{loc})

$$\lim_{k\to 0} u(x,y,k)=u(x,y) \tag{57}$$

and $u(x,y)$ solves (56).

We now prove that the data $u(x, y)$ known for all x, $y \in P$ determine $a_2(x)$ uniquely. Let $w := [1 + a_2(x)]^{1/2} u$, where u solves (56). Then

$$\nabla^2 w - q(x)\, w = -\delta(x-y)[1 + a_2(x)]^{-1/2}, \qquad y \in P, \tag{58}$$

where

$$q(x) := [1 + a_2(x)]^{-1/2} \nabla^2 [1 + a_2(x)]^{1/2}. \tag{59}$$

Since $a_2(x)$ vanishes on P and $y \in P$, one can write (58) as

$$\nabla^2 w - q(x)\, w = -\delta(x-y), \qquad y \in P. \tag{60}$$

Note that $w = u$ on P, so that $w(x, y)$ on P is known. We prove that the data $w(x, y)$ known for all x, $y \in P$ determine $q(x)$ uniquely. If this is done then $\varphi := 1 + a_2(x)$ is obtained as the unique solution to the problem (see (59)):

$$\nabla^2 \varphi - q(x)\, \varphi = 0 \quad \text{in} \quad R^3, \qquad \varphi = 1 \text{ outside } D. \tag{61}$$

It is not hard to see that (61) has at most one solution $\varphi \in H^2_{\mathrm{loc}}$: if φ_1 and φ_2 are solutions to (61) then $h := \varphi_1 - \varphi_2$ solves the problem

$$\nabla^2 h - q(x)\, h = 0, \qquad h = 0 \text{ outside } D. \tag{62}$$

Since $a_2 \in C^2(D)$, the potential q in (62) belongs to $C(D)$. By the unique continuation property for solutions to elliptic equations, (62) imples that $h = 0$ in R^3, so that $\varphi_1 = \varphi_2$.

In order to prove that the data $w(x, y)$, x, $y \in P$, determine $q(x)$ uniquely, we use the argument similar to the one used in the proof of Theorem 2. We start with the equation

$$w - g_0 = \int g_0(x, \xi)\, q(\xi)\, w(\xi, y)\, d\xi, \qquad g_0 = (4\pi\,|x-y|)^{-1}. \tag{63}$$

The left side in (63) is known. Let us denote it by $f(x, y)$. Equation (63) implies that the set of integrals

$$\int q(\xi)\, u(\xi)\, m(\xi)\, d\xi, \qquad u \in N_D(\nabla^2), \qquad m \in N_D(\nabla^2 - q(x)) \tag{64}$$

is known. This is proved in the same way as the similar statement about integrals (48). From the knowledge of the set of the integrals (64) we conclude that $q(x)$ is uniquely determined. This is done as in the proof of Theorem 2. Therefore, we have proved that the data $u(x, y, k=0)$, x, $y \in P$, determine $a_2(x)$ uniquely. The rest of the argument follows the argument in [4]. If $a_2(x)$ is known then the function G, the solution of the problem

$$\nabla^2 G + k^2 G + \nabla \cdot (a_2 \nabla G) = -\delta(x-y) \qquad \text{in } R^3, \tag{65}$$

satisfies the radiation condition at infinity and can be considered known.

Equation (I.7) can be written as

$$u(x, y, k) = G(x, y, k) + k^2 \int G(x, \xi, k)\, a_1(\xi)\, u(\xi, y, k)\, d\xi. \tag{66}$$

For sufficiently small k one proves as in [4] that Eq. (66) is uniquely solvable by iterations and the limit exists

$$f(x, y) := \lim_{k \to 0} k^{-2}(u - G) = \int G_0(x, \xi)\, a_1(\xi)\, G_0(\xi, y)\, d\xi, \qquad x, y \in P, \tag{67}$$

where $G_0 := \lim_{k \to 0} G$ solves problem (56). The fuction $f(x, y)$ can be considered known. The linear equation (67) for the unknown function $a_1(x)$ has at most one solution. This equation for $a_2 = 0$ has been solved in [4] analytically. The proof that Eq. (67) has at most one solution can be given as follows: Consider the homogeneous equation

$$\int G_0(x, \xi)\, a_1(\xi)\, G_0(\xi, y)\, d\xi = 0, \qquad \forall x, y \in P. \tag{68}$$

By the argument used in the proof of Theorem 2 one derives from (68) that

$$\int a_1(\xi)\, u(\xi)\, w(\xi)\, d\xi = 0 \qquad \text{for all} \quad u, w \in N_D(L), \tag{69}$$

where $Lu = \nabla^2 u + \nabla \cdot (a_2 \nabla u) = \nabla \cdot [(1 + a_2) \nabla u] = 0$. Let $u = (1 + a_2)^{-1/2} m(x)$, $w = (1 + a_2)^{-1/2}\, n(x)$. Then $m(x)$ and $n(x)$ satisfy the equation

$$[\nabla^2 - q(x)]\, m = 0 \qquad \text{in } D, \tag{70}$$

where $q(x)$ is given by (59). Equation (69) reduces to

$$\int a_1(\xi)[1 + a_2(\xi)]^{-1}\, m(\xi)\, n(\xi)\, d\xi = 0 \qquad \text{for all} \quad m, n \in N_D(\nabla^2 - q). \tag{71}$$

By Theorem 8, Eq. (71) implies that $a_1(1 + a_2)^{-1} = 0$. Thus $a_1 = 0$. Theorem 3 is proved.

In the next section we give proofs of Theorems 4, 5, and 10.

IV. PROOFS: CONTINUATION

1. *Proof of Theorem* 4. Let us write (I.9) as

$$u(x, y, k) = g(x, y, k)\, h(k) + k^2 \int g(x, \xi, k)\, v(\xi)\, u(\xi, y, k)\, d\xi, \tag{1}$$

where $g = (4\pi\,|x-y|)^{-1} \exp(ik\,|x-y|)$. For $h(k)$ one has the formula

$$\lim_{x \to y} 4\pi\,|x-y|\, u(x, y, k) = h(k), \qquad x, y \in P. \tag{2}$$

Therefore the data $u(x, y, k)$, x, $y \in P$, $k \in R_+ = (0, \infty)$ determine $h(k)$ uniquely and explicitly by formula (2). If $h(k)$ is found then one argues as follows. First assume that $h(0) := A \neq 0$. This assumption will be dropped later. For sufficiently small k, Eq. (1) is uniquely solvable by iterations and the following limit exists [4]:

$$\lim_{k \to 0} k^{-2}(u - gh) = \frac{A}{(4\pi)^2} \int \frac{v(\xi)\, d\xi}{|x-\xi|\,|\xi-y|}, \qquad A = h(0). \tag{3}$$

If $A \neq 0$ we can consider the function

$$f(x, y) := 16\pi^2 A^{-1} \lim_{k \to 0} k^{-2}[u(x, y, k) - g(x, y, k)\, h(k)], \qquad \forall x, y \in P \tag{4}$$

as our data. Then (3) can be written as

$$\int \frac{v(\xi)\, d\xi}{|x-\xi|\,|\xi-y|} = f(x, y), \qquad \forall x, y \in P. \tag{5}$$

The argument which yields Eq. (5) is discussed in detail in [4, Chap. 6], where it was proved that Eq. (5) has at most one solution in $L^2(D)$, and the solution of Eq. (5) was found analytically. Thus we have uniquely and analytically recovered $v(x)$ and $h(k)$ under the assumption $h(0) \neq 0$. Consider the genral case now. Since $h(k)$ is analytic in k according to assumption (I.10), one concludes that $k = 0$ can be zero of $h(k)$ of at most finite order N. Assume that

$$h^{(j)}(0) = 0, \qquad 0 \leqslant j \leqslant N-1, \qquad h^{(N)}(0) \neq 0, \tag{6}$$

so that

$$h(k) = k^N h_N(k), \qquad h_N(0) \neq 0. \tag{7}$$

Let $k^{-N} u := u_N$. Then (1) can be written as

$$u_N(x, y, k) = g(x, y, k)\, h_N(k) + k^2 \int g v u_N\, d\xi. \tag{8}$$

Since $h_N(0) \neq 0$ the above argument is applicable, and the functions $h_N(k)$ and v are uniquely determined by the surface data. Theorem 4 is proved.

Remark 1. If $v(x) = v(x_3)$ then the assumption that v has compact support is not needed and a complete analysis of the inverse problem of finding $v(x_3)$ and $h(k)$ from the surface data is given in [5e].

2. *Proof of Theorem* 5. See [5d, 20, 18], Section V.7, 13. Here the proof is given for $\sigma \in W^{2,\infty}(D)$. Note that if u solves Eq. (I.11) then $w = \sigma^{1/2} u$ solves the equation

$$\nabla^2 w - q(x)\, w = 0 \qquad \text{in } D, \tag{9}$$

where

$$q(x) = \sigma^{-1/2}(x)\, \nabla^2 \sigma^{1/2}(x). \tag{10}$$

If conditions (I.13) and (I.14) hold then $q \in L^\infty(D)$.

Assume first that σ and σ_N are known on Γ. Then the data $\{f, h\}$ for the problem (I.11), (I.12) define the data $\{F, H\}$ for Eq. (9). Here $F := w$ on Γ, $H := w_N$ on Γ, so that

$$F = \sigma^{1/2}(s) f, \qquad H = \sigma^{-1/2}(s)\, h + \tfrac{1}{2}\sigma^{-1/2}(s)\, \sigma_N(s) f. \tag{11}$$

Let us describe the basic steps of the proof.

Step 1. One proves that the data $\{F, H\}$, where F runs through $C^1(\Gamma)$, determine $q(x)$ in Eq. (9) uniquely and constructively. See Section V.7.

Step 2. If $q(x)$ is found then $\sigma^{1/2}(x)$ is uniquely recovered as the solution to the problem

$$\nabla^2 \varphi - q(x)\, \varphi = 0 \qquad \text{in } D, \qquad \varphi := \sigma^{1/2}(x); \tag{12}$$

$$\varphi \text{ and } \varphi_N \text{ on } \Gamma \text{ are know.} \tag{13}$$

The problem (12)–(13) has at most one solution, since it is a Cauchy problem for the elliptic equation (12). If the data φ and φ_N are compatible one can find φ by solving the Dirichlet problem

$$\nabla^2 \varphi - q(x)\ \varphi = 0 \qquad \text{in } D; \qquad \varphi \text{ on } \Gamma \text{ is known.} \tag{14}$$

If zero is not an eigenvalue of the Dirichlet operator $\nabla^2 q(x)$ in D, then problem (14) is uniquely solvable. Otherwise this problem has a finite-parametric family of solutions and there is a uniquely defined solution which has the prescribed value of φ_N.

Step 3. One proves that the set $\{f, \sigma h\}$ determines σ and σ_N on Γ uniquely.

We have already discussed Step 2 in detail. Let us discuss *Step* 1. Pick $\lambda \in \mathbb{C}^3$, $\lambda \cdot \lambda = 0$, multiply Eq. (9) by $\exp(i\lambda \cdot x)$, and integrate over D to get

$$\begin{aligned}\int_D \exp(i\lambda \cdot x)\, q(x)\, w(x)\, dx \\ &= \int_D \exp(i\lambda \cdot x)\, \nabla^2 w\, dx \\ &= \int \left[\exp(i\lambda \cdot s)\, H - F \frac{\partial \exp(i\lambda \cdot s)}{\partial N} \right] ds := a(F, H) \qquad (15)\end{aligned}$$

where a is known since F and H are known.

By Theorem 11 take w of the form

$$w = \exp(iz \cdot x)(1 + R(x, z)), \qquad z \cdot z = 0,\, z \in \mathbb{C}^3, \qquad (16)$$

where $R(x, z)$ satisfies inequality (II.15). Use Lemma 3 in Section III to choose λ and z so that

$$\lambda \cdot \lambda = z \cdot z = 0, \qquad z + \lambda = p, \qquad |z| \to \infty, \qquad |\lambda| \to \infty, \qquad (17)$$

where $p \in R^3$ is an arbitrary fixed vector. Then Eq. (15) gives the Fourier transform of the potential $q(x)$, the function

$$\int_D q(x) \exp(ip \cdot x)\, dx. \qquad (18)$$

If this function is known then $q(x)$ is uniquely determined. This completes *Step* 1. Concerning *Step* 3 we refer to [20] where analytical formulas are obtained for σ and σ_N on Γ given $\{f, \subset \sigma h\}$ $\forall f \in C^1(\Gamma)$. In the paper [6b] it is proved that the set $\{f, \sigma h\}$ determines σ and σ_N on Γ uniquely. In [6b] the kernel of the map $f \mapsto \sigma h$ is assumed known. In practice, a construction of this kernel from the data $\{f, \sigma h\}$ is an ill-posed not simple problem. The basic interest is in recovery of $\sigma(x)$ inside D given the boundary data, and this problem we have solved. In [20] a constructive method is given for finding σ and σ_N on Γ given $\{f, \sigma h\}$ $\forall f \in C^1(\Gamma)$. Theorem 5 is proved. (See also Section V.13.). In [25] and Section V.13 a proof of Theorem 5 is given which does not require the reduction to Schrödinger's equation. It shows that the set $\{f, \sigma h\}$ $\forall f \in C^1(\Gamma)$ determines σ and σ_N on Γ uniquely.

Proof of Theorem 12. This proof has been given in Step 1 of the proof of Theorem 5 for the case $k = 0$. It is the same for $k > 0$.

3. We now prove Theorem 10. The necessity part has been established in Section II. Let us prove the sufficiency part : $C \Rightarrow A \in \mathscr{A}$.

LEMMA 1. *The solution to Eq.* (II.11) *with properties listed in condition C is unique.*

Proof. Suppose there are two (or more) solutions v_j, $j=1, 2$, with the properties *C*. Then $w := v_1 - v_2$ solves the equation

$$w(\theta, k, x) = w(-\theta, -k, x) + \frac{ik}{2\pi}\int_{S^2} A(\theta'', \theta, k)\, w(-\theta'', -k, x)\, d\theta'', \qquad x \in R^3 \quad (19)$$

The function $w(\theta, k, x)$ has asymptotics

$$w = (A_1 - A_2)\, g + o(r^{-1}), \qquad \text{as} \quad r = |x| \to \infty, g := \frac{e^{ikr}}{r}, \qquad (20)$$

where $A_j := A_{q_j}$ are the coefficients in condition (I.3) corresponding to $v_j, j=1, 2$. Note that

$$v(-\theta, -k, x) = A(\theta', -\theta, -k)\, \bar{g} + o(r^{-1}), \qquad r \to \infty. \qquad (21)$$

Here and below the bar stands for complex conjugate. Taking $r = |x| \to \infty$, $xr^{-1} = \theta'$ in (19) and using (20) and (21) one obtains

$$(A_1 - A_2)\, g = B\bar{g} + o(r^{-1}), \qquad r \to \infty, \qquad (22)$$

where

$$B := A_1(\theta', -\theta, -k) - A_2(\theta', -\theta, k) + \frac{ik}{2\pi}\int_{S^2} A(\theta'', \theta, k)[A_1(\theta', -\theta'', -k) - A_2(\theta', -\theta'', -k)]\, d\theta''. \quad (23)$$

It follows from (22) that the coefficient in front of g and $\bar{g}$ should vanish. Thus

$$A_1 = A_2. \qquad (24)$$

Lemma 1 is proved.

The reader can prove as an easy exercise that (22) implies (24) and the equation $B=0$.

LEMMA 2. *The coefficient A_q in* (I.3) *is identical with the given function $A(\theta', \theta) = A(\theta', \theta, k)$, the kernel of equation* (I.11).

Proof. The solution to Eq. (I.11) with properties *C* also satisfies Eq. (I.11) with $A_q(\theta', \theta, k)$ in place of $A(\theta', \theta, k)$. This is well known from direct scattering theory. In terms of the function ψ defined by (I.2) the equations can be written as

$$\psi(\theta, k, x) = S\psi(-\theta, -k, x) \qquad (25)$$

and

$$\psi(\theta, k, x) = S_q\psi(-\theta, -k, x), \qquad (26)$$

where S is the operator with kernel (II.10) and S_q is the operator whose kernel is given by formula (II.10) with A_q in place of A. From (25) and (26) it follows that

$$\int_{S^2} d\theta' [A(\theta', \theta, k) - A_q(\theta', \theta, k)] \, \psi(-\theta', -k, x) = 0 \qquad \forall x \in R^3. \tag{27}$$

Equation (27) implies that $A = A_q$ as follows from Lemma 3 below. Lemma 2 is proved.

LEMMA 3. *Let $f \in L^2(S^2)$ and assume that*

$$\int_{S^2} d\theta \, f(\theta) \, \psi(-\theta, -k, x) = 0 \qquad \forall x \in \Omega_R := \{x : |x| \geqslant R\}. \tag{28}$$

then $f(\theta) = 0$.

Proof. The proof consists of two steps:

Step 1. If (28) holds with $\psi_0 := \exp(ik\theta \cdot x)$ in place of $\psi_- := \psi(-\theta, -k, x)$ then $f = 0$.

Step 2. Equation (28) implies that $f = 0$.

Step 1. Suppose that

$$u(x) := \int_{S^2} d\theta \, f(\theta) \exp(ik\theta \cdot x) = 0 \qquad \forall x \in \Omega_R. \tag{29}$$

Since

$$(\nabla_x^2 + k^2) \, u = 0 \text{ in } R^3, \qquad u = 0 \text{ in } \Omega_R, \tag{30}$$

the unique continuation property for solutions to elliptic equations yields that $u = 0$ in R^3. Therefore the Fourier transform of the distribution with support on S^2 and the density $f(\theta)$ vanishes. Thus $f(\theta) = 0$.

Step 2. It is known from the theory of direct scattering problem that

$$\psi_- = B\psi_0, \tag{31}$$

where B is a bounded linear isomorphism of $C(R^3)$ onto $C(R^3)$ ($B = (I + T)^{-1}$, where $Th := \int g(x, y) \, q(y) \, h(y) \, dy$). Let

$$w(x) := \int_{S^2} d\theta \, f(\theta) \, \psi(-\theta, -k, x). \tag{32}$$

Then

$$[\nabla^2 + k^2 - q(x)] \, w = 0 \text{ in } R^3, \qquad w = 0 \text{ in } \Omega_R. \tag{33}$$

As in Step 1, (33) implies that $w=0$ in R^3:

$$0=\int_{S^2} d\theta f(\theta)\,\psi(-\theta,-k,x) \qquad \forall x\in R^3. \tag{34}$$

Apply operator B^{-1} to (34) to get

$$0=\int_{S^2} d\theta f(\theta)\,B^{-1}\psi=\int_{S^2} d\theta f(\theta)\exp(ik\theta\cdot x). \tag{35}$$

Note that B^{-1} acts on the x variable and therefore can be taken under the sign of the integral in θ. In Step 1 we have proved that (35) implies that $f=0$. Lemma 3 is proved.

The sufficiency part in Theorem 10 is proved. This completes proof of Theorem 10. (Our argument is based on the ideas which appeared for the first time in [2a, b].)

V. Open Problems, Additional Results and Remarks

1. In the proof of Theorems 2 and 3 the question of approximation of the solutions to a homogeneous PDE arises. In [14, Sections 7.3 and 10.5] this question is discussed for general PDE with constant coefficients and it is proved that the closed linear hull in $C^\infty(D)$ of the exponential solutions of the equation $Lu=0$ consists of all its solution in $C^\infty(D)$, where D is a convex compact set. Here L is a general linear differential expression with constant coefficients. An exponential solution of the equation $Lu=\sum_{|\alpha|\leq l} a_\alpha \partial^\alpha u=0$ is the solution of the form $u=\exp(i\lambda\cdot x)\,P(x)$, where $\lambda\in\mathbb{C}^n$ and $P(x)$ is a polynomial. If the polynomial $L(\lambda):=\sum_{|\alpha|\leq l} a_\alpha(i\lambda)^\alpha$ has no multiple factors then one can take $P(x)=1$ and the closed linear hull of the solutions of the form $\exp(i\lambda\cdot x)$ will consist of all solutions of the equation $Lu=0$ in $C^\infty(D)$. The convexity of D is not a restriction in our problems. Indeed, if D is not convex one can consider the closed convex hull of D, denoted ch D. The functions $q(x)$, $v(x)$, $a_j(x)$ that we had in various problems, will vanish outside ch D, and all our arguments can be repeated with ch D in place of D. In these arguments we used just two properties: D is bounded and the function of interest (e.q., q, v, or a_j) vanishes outside of D. If $D\subset R^3_-$ then ch $D\subset R^3_-$. See Section V.6 for another argument.

2. In some of these arguments the requirement that the function of interest is compactly supported can be relaxed. For example, it is sufficient to request that $q(x)$ can decay faster than any exponential in Problems 1 and 7.

3. A list of open problems is given in [4], and another one in [15]. Here we would like to mention some other problems. See also Remark 2 in Section II. One, for which the motivation can be found in [16], is as follows:

(a) Let $D \subset R^3_-$ be a bounded domain, $v(x) \in L^2(D)$, $v = 0$ outside D, and assume that

$$\int_D \frac{v(y)\exp(ik\,|x-y|)}{|x-y|^2}\,dy = 0 \qquad \forall x \in P = \{x : x_3 = 0\}. \tag{1}$$

Here $k > 0$ is a fixed number.

Question. Does (1) imply that $v = 0$?

In [16] the question is answered for $k = 0$. If $k = 0$, the answer is no.

(b) The arguments in Sections III and IV use essentially the analytic properties of the solutions to partial diffeential equations based on the assumption that the functions of interest (q, v, a_j) are compactly supported. To what extent can this assumption be relaxed? For example, can one treat the functions with fall-off rate of the type $O(|x|^{-a})$, $|x| \to \infty$ with some a? In this case the scattering amplitude $A(\theta', \theta, k)$, for example, cannot, generally speaking, be continued analytically in θ, θ'. Therefore our arguments are not directly applicable in this case. For problem 1 with $q = q(|x|) = O(|x|^{-a})$ the uniqueness theorem may not hold [26].

(c) The problem of inverting the backscattering data $A(-\theta, \theta, k)$ given for all $\theta \in S^2$ and all $k \in R_+ = (0, \infty)$ is of interest. For small potentials $q(x)$ this problem was treated in the literature (e.g., in [3, 17]), but global results are not known.

(d) We gave only a partial solution to Problem 6 in Section I. Here much work can be done.

If the polynomial $L_0(z) := \sum_{|\alpha| = l} a_\alpha z^\alpha$ corresponding to the operator (I.15) has no multiple zeros $z \neq 0$, that is, $|\nabla L_0(z)| \neq 0$ for $z \neq 0$, then Condition A in Section II, is satisfied. If $M_0 := \{z : z \in \mathbb{C}^n,\ L_0(z) = 0\}$ then we conjecture that the equation $Lu = 0$, where L is defined in Problem 6, Section I, has a solution $u = \exp(z \cdot x)[1 + R(x, z)]$, where $z \in M$ and $\|R(x, z)\|_{L^2(D_1)} \to 0$ as $|z| \to \infty$, $\operatorname{Im} z \neq 0$. Here $D_1 \subset R^n$ is an arbitrary bounded domain, and it is assumed that the coefficients $b_\alpha(x)$ in (I.16) vanish outside a bounded domain $D \subset R^n$ and are sufficiently smooth: $b_\alpha(x) \in W^{|\alpha|, \infty}(D)$ and the order of $V := L - L_0$ is $\leqslant l - m$, $m \geqslant 2$. If the conjecture is true, then for $L = L_0 + V$ under the above assumptions (I.18) implies that $f = 0$. The proof of this is the same as the proof of Theorem 9, Section II.

(e) How far can one relax the assumption $q(x) \in L^\infty(D)$ in Theorem 11, Section II? In [18] the case $q \in L^2(D)$ is treated.

4. One can use Theorem 12 in Section II in many applications. Here we sketch one of them. See [5d, 7, 8].

Let $q(x) \in L^\infty(D)$, Im $q=0$, $j=1, 2$; $D \subset R^n$ is a bounded domain with a smooth boundary Γ. Let

$$(\nabla^2 + q_j(x) - \lambda_m^{(j)})\, \psi_m^{(j)} = 0 \quad \text{in } D, \qquad \psi_m^{(j)} = 0 \quad \text{on } \Gamma \tag{2}$$

$$\| \psi_m^{(j)} \|_{L^2(D)} = 1. \tag{3}$$

Let

$$\lambda_m^{(1)} = \lambda_m^{(2)} \qquad \text{and} \qquad \psi_{mN}^{(1)} = \psi_{mN}^{(2)} \qquad \text{for all} \quad m = 1, 2, \ldots. \tag{4}$$

The eigenvalues in (2) are ordered so that $\lambda_1^{(j)} < \lambda_2^{(j)} \leqslant \cdots$ and counted according to their multiplicities (since the operator in (2) is selfadjoint, the geometric multiplicities are equal to the algebraic multiplicities). In (4) $\psi_N := \partial\psi/\partial N$ on Γ.

PROPOSITION 1. *If* (4) *holds then* $q_1(x) = q_2(x)$.

Proof. We sketch the proof which is based on Theorem 12 in Section II. Let $q(x) \in L^\infty(D)$, Im $q=0$ and

$$[-\nabla^2 + q(x)]\, G = \delta(x-y) \qquad \text{in } D, \qquad G = 0 \text{ on } \Gamma. \tag{5}$$

We wish to show that the data

$$\{\lambda_m, \psi_{mN}\}, \qquad m = 1, 2, \ldots, \tag{6}$$

where λ_m and ψ_m are the eigenvalues and the normalized eigenfunctions of the Dirichlet operator $-\nabla^2 + q(x)$ in D, determine $q(x)$ uniquely. From this Proposition 1 follows immediately.

Let u solve the problem

$$(-\nabla^2 + q)\, u = 0 \qquad \text{in } D, \qquad u = f \text{ on } \Gamma. \tag{7}$$

Without loss of generality assume that zero is not an eigenvalue of the Dirichlet operator $-\nabla^2 + q$ in D. (If it is, then consider the operator $-\nabla^2 + a + q$, where $a = \text{const}$ chosen so that zero is not an eigenvalue of the new operator.) The solution to (7) is

$$u = -\sum_{m=1}^{\infty} \lambda_m^{-1} f_m \psi_m(x), \tag{8}$$

where

$$f_m := \int_\Gamma f \psi_{mN}\, ds, \tag{9}$$

where the bar stands for complex conjugate, and

$$h := u_N = -\sum_{m=1}^{\infty} \lambda_m^{-1} f_m \psi_{mN}. \tag{10}$$

The last formula is obtained by the formal termwise differentiation of the series (8) along N. Formulas (8) and (10) show that the data (6) uniquely determine the data

$$\{f, h\}. \tag{11}$$

By Theorem 12 in Section II (with $k=0$) one concludes that the data (11) determine $q(x)$ uniquely. Thus the data (4) determine $q(x)$ uniquely.

In order to justify the formal differentiation in (10) note that this differentiation is justified if only a finite number of the coefficients f_m in (8) do not vanish. The set of the finite linear combinations of the functions ψ_{mN} is dense in $L^2(\Gamma)$ [4]. The mapping $\Lambda : f \to h$ is known to be continuous from $H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ for $q \in L^\infty(D)$, while the mapping $\Lambda^{-1} : h \to f$ is continuous from $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$. Thus, if one knows the set $\{f, h\}$ for h running through a dense set in $H^{-1/2}(\Gamma)$ then the corresponding f runs through a dense set in $H^{1/2}(\Gamma)$. Since $C^1(\Gamma) \subset H^{1/2}(\Gamma)$ one knows the set $\{f, h\}$ for all $f \in C^1(\Gamma)$. This is sufficient for the unique recovery of $q(x)$ by Theorem 12. Proposition 1 is proved.

An idea to use formula (10) was used in [7], where the argument is more complicated since Theorem 12 was not used. A relevant paper is also [8], where the method is quite different and the proofs are not given.

5. In Problem 2, Section I, the background refraction coefficient is constant. The case of a variable background when the governing equation is

$$[\nabla^2 + k^2 n_0(x) + k^2 v(x)]\, u = -\delta(x-y) \qquad \text{in } R^3 \tag{12}$$

is treated in [10]. The function $n_0(x)$ in (12), which describes the variable background, is assumed to be known and is fairly general: the basic assumption on $n_0(x)$ is that the Green function G,

$$[\nabla^2 + k^2 n_0(x)]\, G = -\delta(x-y) \qquad \text{in } R^3 \tag{13}$$

has limit as $k \to 0$,

$$G \to (4\pi |x-y|)^{-1} \qquad \text{as} \quad k \to 0. \tag{14}$$

Conditions for (14) to hold are given in [13].

6. In this section we explain why one can take, in the proof of Theorem 1, $\psi = u$ with the properties listed below (III. 42). The reason is that any solution to Equation (II. 13) in D can be approximated with arbitrary accuracy by the scattering solutions (I. 2). We assume that k^2 is not an eigenvalue of the Dirichlet operator $l := \nabla^2 + k^2 - q(x)$ in D, i.e., $k^2 \notin \sigma(l_D)$. This assumption can be made without loss of generality: if it does not hold for D one takes as D a larger domain for which it holds. One can argue as in the proof of Theorem 2 in order to show that the knowledge of $-4\pi A(\theta', \theta, k)$ for all $\theta \in S^2$ implies the knowledge of the integral

$$\int \exp(-ik\theta' \cdot x) \exp(ik\theta \cdot x)(1+R)\, q(x)\, dx \qquad \text{as} \quad |\theta| \to \infty,\, \theta \cdot \theta = 1,\, \theta \in C^3.$$

Multiply (III. 42) by $\varphi(\theta) \in L^2(S^2)$ and integrate over S^2. When $\varphi(\theta)$ runs through $L^2(S^2)$ the set $m(x, k) := \int_{S^2} \varphi(\theta)\, \psi(\theta, k, x)\, d\theta$, where ψ is the scattering solution (I. 2), runs through a dense set in $N_D(l)$. To see this it is sufficient to prove that $m(s, k)$, $s \in \Gamma := \partial D$, runs through a dense set in $L^2(\Gamma)$. Indeed, $m(x, k)$ solves Eq. (II.13) in D and, provided that $k^2 \notin \sigma(l_D)$, if $m(s, k)$ approximates any $L^2(\Gamma)$ fuction then $m(x, k)$ approximates any element of $N_D(l)$. Suppose that there is an $h \in L^2(\Gamma)$ such that $\int_\Gamma dsh(s)$ $m(s, k) = 0\ \forall m$. Then $(*)\ \int_\Gamma dsh(s)\, \psi(\theta, k, s) = 0\ \forall \theta \in S^2$. Define the function $w(x) := \int_\Gamma dsh(s)\, G(x, s)$, where G is the resolvent kernel of l in R^3, that is, $lG = -\delta(x-y)$ in R^3 and G satisfies the radiation condition. Note that

$$G(x, s) = \frac{\exp(ik\,|x|)}{4\pi\,|x|}\, \psi(s, \theta, k) + o(|x|^{-1}) \qquad \text{as} \quad |x| \to \infty,\, x\,|x|^{-1} = -\theta$$

(see [4, p. 46]). Therefore $(*)$ implies that $w = o(|x|^{-1})$ as $|x| \to \infty$. Since $lw = 0$ in $\Omega := R^3 \backslash D$ and $w = o(|x|^{-1})$ one concludes that $w = 0$ in Ω [4, p. 25]. Thus $w = 0$ on Γ. Since $lw = 0$ in D, $w = 0$ on Γ, and $k^2 \notin \sigma(l_D)$, one gets $w = 0$ in D. Therefore $h = 0$. We have proved that the set $\{m(x, k)\}$ is dense in $L^2(D)$ in the set $N_D(l)$. Another proof which does not use the assumption $k^2 \notin \sigma(l_0)$ is given in Sections V. 11–12.

Therefore the knowledge of $A(\theta', \theta, k)$ for all $\theta', \theta \in S^2$ implies the knowledge of the set of integrals $\int dx\, q(x) \exp(-ik\theta' \cdot x)\, u(x, k)$ for all $u \in N_D(l)$, in particular, for u of the form given below formula (III. 42).

7. Here we discuss a point related to Step 1 in the proof of Theorem 5. Suppose that the data $\{f, h\}\ \forall f \in C^1(\Gamma)$ are given, where $f = w$, $h = w_N$ on Γ, w solves Eq. (9). We have shown in Step 3 that these data determine $\tilde{q}(p)$. Suppose there is another potential $q_1(x)$ with the same data. If one argues as in Step 3, then one gets the equation:

$$\int_D dx(qw - q_1 w_1) \exp(i\lambda \cdot x)$$

$$= \int_\Gamma [\exp(i\lambda \cdot s)(w - w_1)_N - (w - w_1)(\exp(i\lambda : s))_N]\, ds. \qquad (15)$$

One takes w as in formula (IV.16) and $w_1 = w$ on Γ. By the assumption, $w_{1N} = w_N$ on Γ. Thus the surface integral in (15) vanishes. If one argues as in the proof of Theorem 5 one concludes that (15) implies after passage to the limit (IV. 17) that $q - q_1 = 0$, so that $q = q_1$. In this argument one uses the fact that the solution w_1 has the form (IV. 16), similar to that of w. Let us prove this. First note that w_1 determined uniquely in D as the solution to the problem

$$(\nabla^2 + k^2 - q_1)\, w_1 = 0 \text{ in } D, \qquad w_1 = w \text{ on } \Gamma \tag{16}$$

extends uniquely to the solution to Eq. (16) in R^3 which equals w in $\Omega := R^3 \setminus D$. This follows from the uniqueness of the solution to the Cauchy problem for elliptic equations: w and w_1 solve the same equation $(\nabla^2 + k^2)\, w = 0$ in Ω (since $q = q_1 = 0$ in Ω) and have the same Cauchy data on Γ. Let us denote the extended function by w_1,

$$w_1 := \begin{cases} w_1 & \text{in } D \\ w & \text{in } \Omega, \end{cases}$$

$$[\nabla^2 + k^2 - q_1(x)]\, w_1 = 0 \qquad \text{in } R^3. \tag{17}$$

Put $w_1 = \exp(iz \cdot x)(1 + R_1)$, $z \cdot z = k^2$. Then

$$LR_1 := (\nabla^2 + 2iz \cdot \nabla)\, R_1 = q_1 R_1 + q_1. \tag{18}$$

Denote by w_2 the solution to (17) of the form (II.14)–(II.15). We claim that $w_1 = w_2$, so that w_1 is of the form (II.14)–(II.15). Indeed, $w_2 = \exp(iz \cdot x)(1 + R_2)$, where R_2 solves Eq. (18). Therefore $\rho := R_1 - R_2$ solves the equation

$$L\rho = q_1 \rho \tag{19}$$

and

$$\rho = L^{-1}\eta \qquad \text{in} \quad \Omega, \qquad \operatorname{supp} \eta \subset D. \tag{20}$$

Equation (20) follows from the fact that both $w = w_1$ in Ω and w_2 in R^3, and in particular in Ω, have R_1 and R_2 of the form (20), where the operator L^{-1} in $L^2(D)$ is defined by the formula (cf. (III.17)).

$$L^{-1}\eta := -\int_{R^3} \frac{\tilde{\eta}(\lambda) \exp(i\lambda \cdot x)}{\lambda^2 + 2\lambda \cdot z}\, d\lambda, \qquad z \cdot z = k^2. \tag{21}$$

This is the operator used in the proof of Lemma III.5.

LEMMA A. *The only solution to* (19)–(20) in $L^2_{\text{loc}}(R^3)$ *is* $\rho = 0$.

Proof. It follows from (19) that

$$\rho = L^{-1} q_1 \rho + \alpha, \tag{22}$$

where α is a solution in $L^2_{\text{loc}}(R^3)$ of the equation

$$L\alpha = 0 \qquad \text{in } R^3. \tag{23}$$

It follows from (20) and (22) that

$$\alpha = L^{-1}\eta_1 \qquad \text{in} \quad \Omega, \qquad \operatorname{supp} \eta_1 \subset D. \tag{24}$$

Thus

$$\alpha = L^{-1}\eta_1 + \varphi \text{ in } R^3, \qquad \varphi \in L^2_{\text{loc}}, \qquad \operatorname{supp} \varphi \subset D. \tag{25}$$

From (23) and (25) one gets

$$L(\lambda)\, \tilde{\alpha} = 0, \qquad L(\lambda) := \lambda^2 + 2\lambda \cdot z \tag{26}$$

$$\tilde{\alpha} = L^{-1}(\lambda)\, \tilde{\eta}_1 + \tilde{\varphi}, \tag{27}$$

where the tilde denotes the Fourier transform. One concludes from (26) that

$$\tilde{\alpha} = 0 \qquad \text{if} \quad L(\lambda) \neq 0. \tag{28}$$

Let $M := \{\lambda : \lambda \in R^3, L(\lambda) = 0\}$. Choose a point $\lambda_0 \in M$ and let $\lambda \notin M$, $\lambda \to \lambda_0$, in (27). Since $\tilde{\varphi}$ is an entire function of λ one concludes from (27) and (28) that there exists a finite limit

$$\lim_{M \not\ni \lambda \to \lambda_0} L^{-1}(\lambda)\, \tilde{\eta}_1(\lambda) \tag{29}$$

Since $\tilde{\eta}_1$ is an entire function of λ one concludes that $\tilde{\eta}_1(\lambda) = L(\lambda)\, \beta(\lambda)$, where $\beta(\lambda)$ is also entire. Therefore $\tilde{\alpha}$ is an entire function which vanishes outside M. Since M is a line one concludes that $\tilde{\alpha}$ vanishes on an open set and therefore $\tilde{\alpha} \equiv 0$ and $\alpha = 0$. The lemma is proved.

Another, a shorter but less constructive way to prove that the knowledge of $\{f, h\}$ $\forall f \in C^1(\Gamma)$ determines $q(x)$ uniquely is as follows (see also Section V.11, where the same idea is used). If $l_j u_j = 0$, $j = 1, 2$, $l_j = \Delta + k^2 - q_j(x)$ and the sets $\{f, h\}$ are the same for q_1 and q_2, then (*) $l_1 u = q u_2$, where $q := q_1 - q_2$, $u := u_1 - u_2$. Multiply (*) by an arbitrary $w_1 \in N_D(l_1)$, integrate over D and then by parts to get $0 = \int_D q u_2 w_1\, dx$. Here we took $u_2 = u_1$ on Γ and used the basic assumption which says that $u_2 = u_1$ on Γ implies $u_{2N} = u_{1N}$ on Γ. Since the set $\{u_2 w_1\}$ $\forall u_2 \in N_D(l_2)$, $\forall w_1 \in N_D(l_1)$ is complete in $L^2(D)$ by Theorem 9, it follows that $q = 0$, i.e., $q_1(x) = q_2(x)$.

See also Section V.13 for a similar argument for Problem 5.

8. In this section *we assume that there are two potentials $q_j(x)$, $j = 1, 2$ with the same scattering amplitude for a fixed $k > 0$, $A_1(\theta', \theta, k) = A_2(\theta', \theta, k) := A(\theta', \theta, k)$.*

We wish to prove that this implies that $\tilde{q}_1 = \tilde{q}_2$, so that $q_1(x) = q_2(x)$.

If one argues as in the proof of Theorem 1 and in Section V.6, then one gets the equation

$$0=\int dx \exp(-ik\theta'\cdot x)(q_1 u_1 - q_2 u_2). \tag{30}$$

Here

$$u_1 = \exp(ik\theta\cdot x)\ (1+R_1) \tag{31}$$

$$\| R_1 \|_{L^2(D)} \to 0 \qquad \text{as} \quad |\theta| \to \infty, \theta\cdot\theta = 1, \theta\in\mathbb{C}^3 \tag{32}$$

$$u_1 = \lim_{\varepsilon\to 0}\int_{S^2} \psi_1(x,\theta,k)\, h_\varepsilon(\theta)\, d\theta, \tag{33}$$

and ψ_1 is the scattering solution (I.1)–(I.3) for $q=q_1$:

$$l_1\psi_1 := (\nabla^2 + k^2 - q_1)\,\psi_1 = 0 \qquad \text{in } R^3, \tag{34}$$

and u_2 is defined as

$$u_2 = \lim_{\varepsilon\to 0}\int_{S^2} \psi_2(x,\theta,k)\, h_\varepsilon(\theta)\, d\theta. \tag{35}$$

In (33) and (35) lim means limit in $L^2(D_1)$, where $D_1 \subset R^3$ is an arbitrary compact domain and u_1 is a linear combination of the scattering solutions. Note that since $A_1 = A_2$ one has

$$\psi_1 = \psi_2 \qquad \text{in} \quad \Omega = R^3\backslash D. \tag{36}$$

This follows from the uniqueness theorem of Rellich's type (see [4, p. 25]). If

$$(\nabla^2 + k^2)\,\omega = 0 \qquad \text{in} \quad \Omega, k>0,\ \omega = o(|x|^{-1}),\ |x|\to\infty \tag{37}$$

then $\omega = 0$ in Ω.

The function $\omega = \psi_1 - \psi_2$ solves (37) and therefore (36) holds. Thus

$$u_1 = u_2 \qquad \text{in } \Omega. \tag{38}$$

Let φ_2 denote the solution to the equation

$$l_2\varphi_2 := (\nabla^2 + k^2 - q_2)\,\varphi_2 = 0 \qquad \text{in } R^3 \tag{39}$$

which has the form (31)–(32) with R_2 in place of R_1. We claim that

$$\varphi_2 = u_2 \qquad \text{in } R^3, \tag{40}$$

so that u_2 has the form (31)–(32) with R_2 in place of R_1. To prove (40) note that u_2 solves (39). Define r by the formula

$$u_2 = \exp(ik\theta\cdot x)(1+r). \tag{41}$$

Then r and R_2 solve the same equation,

$$LR := (\nabla^2 + 2ik\theta \cdot \nabla) R = q_2 R + q_2, \tag{42}$$

so that $\rho := r - R_2$ solves the equation

$$L\rho = q_2 \rho. \tag{43}$$

Furthermore (38) implies that ρ satisfies (20). By Lemma A, $\rho = 0$. Thus (40) holds. Pass to the limit

$$|\theta| \to \infty, \qquad |\theta'| \to \infty, \qquad \theta \cdot \theta = \theta' \cdot \theta' = 1, k(\theta - \theta') = p \tag{44}$$

in (30), to get

$$\tilde{q}_1(p) - \tilde{q}_2(p) = 0. \tag{45}$$

Here $p \in R^3$ is arbitrary. Thus $q_1 = q_2$. See also [21] and Section V.11.

9. In this section we point out that Problem 1 can be reduced to the problem of finding $q(x)$ from the knowledge of the set $\{f, h\}$ $\forall f \in C^1(\Gamma_R)$. Here Γ_R: $\{x: |x| = R\}$, $x \in R^3$, and

$$lu := (\nabla^2 + k^2 - q(x))\, u = 0 \qquad \text{in} \quad B_R := \{x: |x| \leqslant R\} \tag{46}$$

$$u = f, \qquad u_N = h \qquad \text{on } \Gamma_R. \tag{47}$$

and we assume that $D \subset B_R$.

Indeed, if $A(\theta', \theta, k)$ is known for all θ', $\theta \in S^2$, and a fixed $k > 0$, then one can uniquely find $\psi(\theta, k, x)$, the scattering solution defined in (I.1)–(I.3), in the region $\Omega_R := R^3 \backslash B_R$. In fact, (see [24]):

$$\psi = \exp(ik\theta \cdot x) + \sum_{n=0}^{\infty} A_n Y_n(\theta')\, h_n(kr), \qquad r \geqslant R, \tag{48}$$

where $r = |x|$, $\theta' = r^{-1}x$, $Y_n(\theta')$ is the orthonormal in $L^2(S^2)$ system of spherical functions,

$$A_n := A_n(\theta, k) := \int_{S^2} A(\theta', \theta, k)\, \overline{Y_n(\theta')}\, d\theta' \tag{49}$$

and $h_n(z)$ is the spherical Hankel function normalized so that

$$h_n(r) \sim r^{-1} \exp(ir) \qquad \text{as} \quad r \to \infty. \tag{50}$$

One obtains ψ_N on Γ_R by differentiating (48). As was proved in Section V.6, the closure of the linear span of the scattering solutions $\{\psi(\theta, k, x)\}$ $\forall \theta \in S^2$ is dense in $L^2(D)$ in the set of all H^2 solutions to Eq. (46). Outside D, in particular in a neighborhood of Γ_R, any solution to (46) is in the closure of the span $\{\psi(\theta, k, x)\}$ $\forall \theta \in S^2$. Therefore, if one knows the mapping

$\psi|_{\Gamma_R} \to \psi_N|_{\Gamma_R}$, one knows the mapping $u|_{\Gamma_R} \to u_N|_{\Gamma_R}$ for any solution to (46). Knowledge of this map allows one to recover $q(x)$ uniquely. This was proven in Step 1 in the proof of Theorem 5 (for $k=0$, but the argument is similar for $k>0$).

This suggests a numerical method for recovery of a compactly supported $q(x)$ from the scattering amplitude $A(\theta', \theta, k)$ known for all θ', θ, and fixed $k>0$. The method consists of the following steps. First, given $A(\theta', \theta, k)$ find the map $u \mapsto u_N$ on Γ_R using (48) and taking into account that if

$$u = \lim_{\varepsilon \to 0} \int_{S^2} h_\varepsilon(\theta)\, \psi(\theta, k, x)\, d\theta, \qquad x \in \Gamma_R \tag{51}$$

then

$$u_N = \lim_{\varepsilon \to 0} \int_{S^2} h_\varepsilon(\theta)\, \psi_N(\theta, k, x)\, d\theta \qquad x \in \Gamma_R. \tag{52}$$

In particular, if one takes as u the solution (II.14)–(II.15) then one obtains by the formula analogous to (IV.15) the Fourier transform (IV.18) of $q(x)$. Therefore, the basic numerical problem is to choose the linear combination (51) so that on Γ_R u takes the values of the function (II.14–II.15). One can try to choose h_ε, or more generally, $d\mu_\varepsilon(\theta)$, where $\mu_\varepsilon(\theta)$ is a signed measure on S^2 (this allows one to use not only $h_\varepsilon \in L^2(S^2)$ but also h_ε which contains the delta function components), such that, e.g.,

$$\left\| \exp(-ik\theta \cdot x) \int_{S^2} \psi(\alpha, k, x)\, d\mu(\alpha) - 1 \right\|_{R, R_1} = \min, \tag{53}$$

where $\| f \|^2_{R,R_1} := \int_{B_{R,R_1}} |f|^2 (1+|x|)^{-3/2}\, dx$, $B_{R,R_1} = B_{R_1} \backslash B_R$, minimum is taken over all measures $\mu(\alpha)$ on S^2, $\theta \cdot \theta = 1$, $\theta \in C^3$, $|\theta| \gg 1$ is fixed, $k(\theta - \theta') = p$, where $p \in R^3$ is a given vector, $R_1 = \varepsilon^{-1}$.

From (II.15) it follows that min in (53) is $o(1)$ as $|\theta| \to \infty$, $\theta \cdot \theta = 1$.

If a sequence $\mu_{\delta,\varepsilon}$ is minimizing for (53) as $\delta \to 0$, then there is a $\delta = \delta(\varepsilon)$ such that as $\varepsilon \to 0$ the sequence $\mu_\varepsilon(\alpha) = \mu_{\delta(\varepsilon),\varepsilon}$ generates

$$u_\varepsilon := \int_{S^2} \psi(\alpha, k, x)\, d\mu_\varepsilon(\alpha) \tag{54}$$

which can be used as in the proof of Theorem 5 to obtain the value $\tilde{q}(p)$ as in (IV.18). Namely [19],

$$\begin{aligned} \tilde{q}(p) &:= \int dx\, q(x) \exp(ip \cdot x) \\ &= -4\pi \lim_{\substack{|\theta| \to \infty,\, \theta \cdot \theta = 1 \\ \theta - \theta' = p}} \left\{ \lim_{\varepsilon \to 0} \int_{S^2} A(\theta', \alpha)\, d\mu_\varepsilon(\alpha) \right\}. \end{aligned} \tag{55}$$

In [19] and [21] one can find a complete justification of formula (55).

10. An argument similar to the one given in 7 and 8 can be used in the proof or Theorem 2 to discuss the case when v_j, $j=1, 2$, produce the same surface data. One can also reduce the proof to the proof of Theorem 1. Indeed the surface data uniquely determine the right side of (III.46) for all $x, y \notin D$. Passing to the limits $|x| \to \infty$, $x|x|^{-1} = \theta'$, $|y| \to \infty$, $y|y|^{-1} = -\theta$ one obtains the scattering amplitude for the potential $q(x) = -k^2 v(x)$. Indeed,

$$g(x, \xi) = (4\pi |x|)^{-1} \exp(ik|x|) \exp(-ik\theta' \cdot \xi)(1 + O(|x|^{-1})),$$

$$u(\xi, y) = (4\pi |y|)^{-1} \exp(ik|y|) \psi(x, \theta, k)(1 + O(|y|^{-1})).$$

Thus, if v_1 produces the same surface data as v_2 then $u_1(\xi, y, k) = u_2(\xi, y, k)$ for all $y \notin D$, where u_j corresponds to v_j and stands in (III.46)) in place of u. A numerical method for solving problem 2 is given in [23].

11. A simple way to see that completeness of the products $\{w_j u_j\}$, $\forall w_j \in N(L_1)$, $\forall u_j \in N(L_2)$ implies uniqueness of the solution to inverse problems we illustrate using Problem 2. The same argument holds for Problem 1. Suppose that $L_j = \Delta + k^2 + k^2 v_j(x)$, and the surface data $u_j = u_j(x, y, k)$ $\forall x, y \in P = \{x : x_3 = 0\}$ are the same for v_j, $j = 1, 2$. Then $\partial u_1/\partial x_3 = \partial u_2/\partial x_3$ on P because $u_1 = u_2$ in the half-space $x_3 > 0$ (being solutions to Helmholtz's equation with the same Dirichlet data on P and satisfying the radiation condition). One has $(*)$ $L_1 u = k^2 v u_2$, where $L_1 u := \Delta u + k^2 u + k^2 v_1 u$, $u := u_1 - u_2$, $v := v_2 - v_1$, $u = \partial u/\partial x_3 = 0$ on P. By the uniqueness of the solution to the Cauchy problem one has $u = 0$ outside the support of v, i.e., outside D. Multiply $(*)$ by an arbitrary $w_1 \in N_D(L_1)$ and integrate by parts using $u = \partial u/\partial N = 0$ on $\Gamma = \partial D$ to get $0 = \int_D v u_2 w_1 \, dx$ for all $w_1 \in N_D(L_1)$ and all $u_2 \in N_D(L_2)$. *Since the set* $\{uw\}$ $\forall u \in N_D(L_2)$, $\forall w \in N_D(L_1)$ *is complete in* $L_2(D)$ *one concludes* $v = 0$. Note that the set $u_2(x)|_D$ is complete in $N_D(L_2)$ in $L^2(D)$ if $u_2(x) \in N_D(L_2)$ runs through the subset of scattering solutions in $N_D(L_2)$; that is, $L_2 u_2 = 0$, $u_2 = \exp(ik\theta \cdot x) + \phi$, where ϕ satisfies the radiation condition. To see this, assume that $f \in N_D(L_2)$ and $(**)$ $\int_D f u_2 \, dy = 0$ for all $u_2 \in N_D(L_2)$ which are scattering solutions. Define $w(x) := \int_D G f \, dy$, where G is Green's function of L_2 satisfying the radiation condition, $L_2 G = -\delta(x - y)$ in R^3. The condition $(**)$ implies that $w = O(|x|^{-2})$ as $|x| \to \infty$ (because $G = (\exp(ik|x|)/4\pi |x|) u_2(y, \theta, k) + O(1/|x|^2)$ as $x/|x| = -\theta$, $|x| \to \infty$ [4, p. 46]). Since $(\Delta + k^2) w = 0$ in $\Omega = R^3 \backslash D$, $w = O(|x|^{-2})$ as $|x| \to \infty$, one has $w = 0$ in Ω ([4, p. 25]). Thus $w = w_N = 0$ on $\Gamma = \partial D$. Furthermore, $L_2 w = -f$ in D. Multiply this equation by $\bar{f}$, integrate by parts, use $L_2 f = 0$ in D and $w = w_N = 0$ on Γ, to get $0 = \int_D |f|^2 \, dx$. Thus $f = 0$.

12. One can prove that the set $\{\psi(\theta, k, x)\}$ $\forall \theta \in S^2$, $k > 0$ is fixed, is complete in $H^1(D_1)$ in $N_{D_1}(l_q) := \{u : l_q u = \Delta u + k^2 u - q(x) u = 0 \text{ in } D_1\}$, where D_1 is an arbitrary compact domain in R^3. Indeed, let $l_q f = 0$, $\int_{D_1}(\psi f + \nabla\psi\nabla\bar{f})\,dx = 0$. Then $\int_{D_1}\psi(\bar{f} - \Delta\bar{f})\,dx + \int_{\Gamma_1}\psi(\partial\bar{f}/\partial N)\,ds = 0$. Define $\phi(x) := \int_{D_1} G(x, y)(\bar{f} - \Delta\bar{f})\,dy + \int_{\Gamma_1} G(x, s)(\partial\bar{f}/\partial N)\,ds$. Then $\phi = 0$ in Ω_1, $\phi = 0$ on Γ_1, $\phi_N^- = 0$ on Γ_1, ϕ_N^- denotes the normal derivative from outside D_1, $\phi_N^+ - \phi_N^- = \partial\bar{f}/\partial N$, so $\phi_N^+ = \partial\bar{f}/\partial N$, $l_q G = -\delta(x-y)$. One has $l_q\phi = -\bar{f} + \Delta\bar{f}$. Multiply this equation by f, integrate over D_1 and then by parts, to get $-\int_{D_1}(|f|^2 + |\nabla f|^2)\,dx = 0$. Thus $f = 0$.

13. Let us sketch a new proof of Theorem 5. Suppose σ_1 and σ_2 produce the same data (I.12), i.e., $\{f, \sigma_j u_{jN}\}$ on Γ, $j = 1, 2$. We wish to prove that this implies $\sigma_1 = \sigma_2$ in D. Subtract from Eq. (I.11) with $\sigma = \sigma_1$, $u = u_1$ this equation with $\sigma = \sigma_2$, $u = u_2$, to get $(*)\ \nabla\cdot(\sigma_1(x)\nabla u) = -\nabla\cdot(\sigma(x)\nabla u_2)$, where $u := u_1 - u_2$, $\sigma := \sigma_1 - \sigma_2$. Multiply $(*)$ by an arbitrary solution w_1 of the equation $\nabla\cdot\sigma_1\nabla w_1 = 0$ in D, and integrate over D and then by parts, to get

$$\int_\Gamma (w_1\sigma_1 u_N - u\sigma_1 w_{1N})\,ds = \int_D \sigma(x)\nabla u_2\nabla w_1\,dx - \int_\Gamma \sigma(x)\,u_{2N} w_1\,ds.$$

Choose u_2 so that $u_2 = u_1$ on Γ. Then the above equation yields

$$\int_D \sigma(x)\nabla u_2\cdot\nabla w_1\,dx = \int_\Gamma (\sigma_1 w_1 u_{1N} - \sigma_1 w_1 u_{2N} + \sigma_1 w_1 u_{2N} - \sigma_2 u_{2N} w_1)\,ds$$

$$= \int_\Gamma w_1(\sigma_1 u_{1N} - \sigma_2 u_{2N})\,ds = 0.$$

One can prove, using Theorem 11 in Section II and assuming $\sigma \in H^3(D)$, that the set $\{\nabla w_1\cdot\nabla u_2\}$, $\forall w_1 \in N_D(l_{\sigma_1})$, $\forall u_2 \in N_D(l_{\sigma_2})$, is complete in $L^2(D)$. Here $N_D(l_\sigma) := \{u : l_\sigma u = 0 \text{ in } D,\ u \in H^2(D)\}$, $l_\sigma u := \nabla\cdot(\sigma(x)\nabla u)$. Therefore, the equation $\int_D \sigma\nabla w_1\cdot\nabla u_2\,dx = 0$ $\forall w_1 \in N_D(l_{\sigma_1})$, $\forall u_2 \in N_D(l_{\sigma_2})$ imply that $\sigma = 0$; i.e., $\sigma_1(x) = \sigma_2(x)$ in D. The assumption $\sigma \in H^3(D)$ is made for technical reasons and can be relaxed. If $\sigma \in W^{3,\infty}$ then function R in (II.14) admits the estimate $\|\nabla R\|_{L^2(D_1)} \leqslant c\,|z|^{-1/2}$ as $|z| \to \infty$, $z\cdot z = k^2$, which is used in the proof of completeness of the set $\{\nabla w_1\cdot\nabla u_2\}$. See [25].

14. Let us assume that in Theorem 12 the mapping $T: f \to h$, is given, and $0 \notin \sigma_D(l)$, that is 0 is not an eigenvalue of the Dirichlet operators $l_0 = \Delta + k^2$ and $l := \Delta + k^2 - q(x)$ in D. We want to construct the solution (II.14–II.15). This solution solves the equation $(*)\ u = u_0 + Qu$, $u_0 := \exp(ik\theta\cdot x)$, $\theta\cdot\theta = 1$, $\theta \in C^3$, $Qu := -\int Gqu\,dy$, where $(\Delta + k^2)\,G = -\delta(x-y)$ in R^3, $G(x) = \exp(ik\theta\cdot x)\,G_0(x)$, $G_0(x) = (2\pi)^{-3}\int \exp(i\lambda\cdot x)(\lambda^2 + 2k\theta\cdot\lambda)^{-1}\,d\lambda$, $\int = \int_{R^3}$. In $\Omega := R^3\backslash D$ one can write $Qu = Sv := \int_\Gamma G(x, s)\,v(s)\,ds$, since Qu and Sv solve the same equation $(\Delta + k^2)\,u = 0$ in Ω and have the same behavior at infinity. See Section V.16. Here $v = S_0^{-1}(Qu|_\Gamma)$, $S_0 v := Sv|_\Gamma$. The operator S_0 is an isomorphism of $H^m(\Gamma)$ onto $H^{m+1}(\Gamma)$ if $0 \notin \sigma_D(l_0)$ [4, p. 81]. Equation $Tu = u_N$ can be written as $T(u_0 + Sv) = u_{0N} + (Av - v)/2$ [4, p. 14], where $Av := 2\int_\Gamma (\partial G(s,t)/\partial N_s)\, v\, dt$, we took $\partial(Sv)/\partial N^-$, since $Qu = Sv$ in Ω, and $\partial/\partial N^-$ denotes the limit value of the normal derivative on Γ from Ω. Thus (**) $v - Av + 2TSv = 2u_{0N} - 2Tu_0$. This equation for v is solvable since (*) is solvable and S_0 is an isomorphism. If $|\theta| \gg 1$, $\theta \cdot \theta = 1$, then the solution is unique since $TSv = (Av - v)/2$ implies (***) $w = -\int Gqw\, dy$ in R^3, $w = Sv$ in Ω. Equation (***) implies $w = 0$ if $|\theta| \gg 1$, $\theta \cdot \theta = 1$. Indeed, put $w = \exp(ik\theta \cdot x)\, r$. Then (***) becomes $r = -\int G_0 qr\, dy$, and $\|r\|_{L^\infty(D)} \leqslant \eta \|q\|_{L^2(D)} \|r\|_{L^\infty(D)}$, where $\eta \to 0$ as $|\theta| \to \infty$, $\theta \cdot \theta = 1$ (see [18]). Thus $r = 0$ if $|\theta| \gg 1$. Therefore $w = 0$. Since S_0 is invertible this implies $v = 0$. Therefore the solution (II.14)–(II.15) is given in Ω by the formula $u = u_0 + Sv$, where v is the unique solution to (**) if $|\theta| \gg 1$, $\theta \cdot \theta = 1$. Computing T from the set $\{f, h\}$ is an ill-posed problem. The results in [20] can be helpful in solving this problem.

Let us explain why $TSv = (Av - v)/2$ implies (***). Let w solve the equation $l_q w = 0$ in R^3. Then $w = -\int_D Gqw dy + w_0$, where $l_0 w_0 = 0$ in R^3. Let $Sv = w$ on Γ. Then $Tw = w_N^+$ by the definition of T, where w_N^+ is the limit value of w_N on Γ from D. Since $w \in H^2_{\text{loc}}$ one has $w_N^+ = w_N^-$. Thus $w_N^- = (Av - v)/2 = (Sv)_N^-$. Thus $w = Sv$ on Γ, $w_N^- = (Sv)_N^-$ on Γ, $l_0 w = 0$ in Ω, $l_0 Sv = 0$ in Ω. By the uniqueness of the solution to the Cauchy problem for elliptic equations, $w = Sv$ in Ω. Thus $-\int_D Gqwdy + w_0 = Sv$ in Ω. This implies that (†) $w_0 = Sv + \int_D Gqwdy$ in Ω. Since $l_0 w_0 = 0$ in R^3, condition (†) is a boundary condition at infinity which implies that $w_0 = 0$ by an argument similar to the one given in Section V.7. Namely, if $w_0 = \exp(ik\theta \cdot x)\phi$, then $L\phi := (\Delta\phi + 2ik\theta \cdot \nabla)\phi = 0$, $\phi = \int_\Gamma G_0 \mu ds + \int_D G_0 q w_1\, dy$ in Ω, $\mu := \exp(-ik\theta \cdot y)\, v(y)$, $w_1 := \exp(-ik\theta \cdot y)\, w(y)$. Thus $L(\lambda)\tilde{\phi} = 0$ and $\tilde{\phi} = L^{-1}(\lambda)\, \tilde{h}(\lambda)$, where $L(\lambda) := \lambda^2 + 2k\theta \cdot \lambda$ and $\tilde{h}(\lambda)$ is an entire function of λ. Thus $\tilde{\phi}(\lambda)$ is meromorphic and vanishes if $\lambda \notin \{\lambda : L(\lambda) = 0\}$. In particular $\tilde{\phi}(\lambda)$ vanishes in a ball in R^3. Therefore $\tilde{\phi} \equiv 0$, and $\phi = 0$. Thus $w_0 = 0$, and (***) holds.

Let us finally prove that equation (**) is of Fredholm's type in $L^2(\Gamma)$. Indeed $TSv = \partial w/\partial N^+$, where $w = -\int_D Gqwdy + \int_\Gamma (GTSv - G_N Sv)\, ds$. Thus

$$\frac{\partial w}{\partial N^+} = -\frac{\partial}{\partial N^+}\int_D G(x,y)q(y)FSvdy + \frac{ATSv + TSv}{2} - \frac{\partial}{\partial N^+}\int_\Gamma G_N Svds. \tag{E}$$

Here $F: H^l(\Gamma) \to H^{l+1/2}(D)$ is an operator which solves the problem $l_q w = 0$ in D, $w = f$ on Γ, $Ff = w$, A is the operator of the potential theory $Af := 2\int_\Gamma (\partial G(s,t)/\partial N_s)\ f(t)\, dt$ (see [4, p. 14]). If $v \in H^l(\Gamma)$ then $Sv \in H^{l+1}(\Gamma)$, $FSv \in H^{l+3/2}(D)$, $\int_D Gfdy \in H^2(D)$ if $f \in L^2(D)$, $(\partial/\partial N^+) \int_D Gfdy \in H^{1/2}(\Gamma)$. Thus the operator $(\partial/\partial N^+) \int_D GqFSvdy$ is compact in $H^l(\Gamma)$, $0 \leqslant l \leqslant \frac{1}{2}$, compactness for $l = \frac{1}{2}$ follows from the above argument and from compactness of the embedding $H^{l+3/2}(D) = H^2(D) \to L^\infty(D)$. It is known that the operator $A: L^2(\Gamma) \to L^2(\Gamma)$ is compact [4]. The operator $(\partial/\partial N^+)$ $\int_\Gamma G_N Sv ds$ is bounded in $L^2(\Gamma)$ (see [4]). One has by Green's formula $\int_\Gamma G_N Sv ds = \int_\Gamma G(Sv)_N^+ ds - Sv$. Thus

$$\frac{\partial}{\partial N^+} \int_\Gamma G_N Sv ds = \frac{A(Sv)_N^+ + (Sv)_N^+}{2} - \frac{Av + v}{2}$$
$$= A(Av + v)/4 - \frac{Av + v}{4} = (A^2 v - v)/4.$$

Since $0 \notin \sigma_D(l_0)$ the operator $I - A$ is invertible in $L^2(\Gamma)$. Since A is compact in $L^2(\Gamma)$, $(I - A)^{-1} = I + A_1$, where A_1 is also compact. Therefore equations $TSv = \partial w/\partial N^+$ and (E) imply $TSv = Bv + \frac{1}{2}v$, where B is compact in $L^2(\Gamma)$. Thus, Eq. (**) is of the form $v + B_1 v = f$ where B_1 is compact. Therefore (**) is a Fredholm equation in $L^2(\Gamma)$. We have proved earlier that the homogeneous equation (**) has only the trivial solution. Thus (**) can be solved numerically by a projection method [4, p. 154].

15. Uniqueness theorems for Problems 1 and 2 (IP1 and IP2) remain valid for complex-valued $q(x)$ and $v(x)$ provided that $k > 0$ is a point at which the Green functions of the operators (I.1) and (I.5) satisfy the limiting absorption principle. In other words, the integral equation $\mathscr{G} = g - \int gq\mathscr{G} dy$ and a similar equation for problem (I.5) are uniquely solvable. Here $\mathscr{G}$ is the kernel of the operator l_q^{-1}, where l_q is given by (I.1), and $\mathscr{G}$ satisfies the radiation condition, $g := (4\pi |x - y|)^{-1} \exp(ik |x - y|)$.

16. LEMMA B. *If* $Qf = Sv$ *on* Γ, $f \in L^2(D)$, $\operatorname{supp} f \subset D$, *and* $k^2 \notin \sigma_D(l_0)$, *then* $Qf = Sv$ *in* Ω.

Proof. Let $u := Qf$, $w := Sv$. Green's formula yields

$$u(x) = \int_{|s| = R} [u_N(s) G(x, s) - u(s) G_N(x, s)] ds - \int_\Gamma [\cdots] ds,$$

where $\cdots$ denotes the same integrand as in the first integral, u_N is the normal derivative, and $R > 0$ is a large number, so that $D \subset B_R := \{y : |y| \leqslant R\}$. One has

$$\int_{|s| = R} [\cdots] ds = \int_D d\xi f(\xi) \int_{|s| = R} [G_N(s, \xi) G(x, s) - G(s, \xi) G_N(x, s)] ds$$
$$= \int_D d\xi f(\xi) \int_{B_R} [G(y, \xi) \delta(x - y) - G(x, y) \delta(y - \xi)\} dy = 0.$$

Thus, if $u = Qf$, $\operatorname{supp} f \subset D$, $f \in L^2(D)$, then $u(x)$ in Ω is given by the formula

$$u(x) = -\int_\Gamma [u_N(s) G(x, s) - u(s) G_N(x, s)] ds, \qquad x \in \Omega. \tag{$*$}$$

Similar formula holds for w. Define $v := u - w$. Subtract formula $(*)$ for w from $(*)$ and use $v = 0$ on Γ to get

$$v(x) = -\int_{\Gamma} v_N(s)\, G(x, s)\, ds, \qquad x \in \Omega. \tag{$**$}$$

Let $x \in \Gamma$ in $(**)$. Then $\int_\Gamma v_N(s)\, G(t, s)\, ds = 0$, $t \in \Gamma$. Since $k^2 \notin \sigma_D(l_0)$, it follows that $v_N(s) = 0$. Therefore $v = v_N = 0$ on Γ, $l_0 v = 0$ in Ω. By the uniqueness of the solution to the Cauchy problem one concludes that $v = 0$ in Ω. Lemma B is proved.

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